On Approximation of Stepfunctions

W.A.Deuber
Bielefeld University

A.V.Rasskazov, A.A.Sapozhenko
Moscow State University

1 Introduction

Applications in VLSI modeling (see [1]) suggest the following problem. The results were partially announced in [2]. Let L_n be graph of the decreasing unit stepfunction with "corners" $(i, n-i+1)$ for $i = 1, ..., n$. The problem is to find good approximations of L_n by stepfunctions G with less than n corners. The following measures for the quality of approximation seem to be relevant.

- $s(G)$: the area between curves G and L_n (area of symmetric difference of G and L_n).
- $r(G)$: the area of the largest rectangle containing in the figure between the curves G and L_n.

Let $K \oplus M$ be the region bounded by stepfunctions K and M. A stepfunction G^* with less than n corners is called *s-optimal* (*r-optimal*) if it minimizes the area of the region $L_n \oplus G$ where G is a stepfunction with less than n corners (respectively, the area of the largest rectangle, contained in $L_n \oplus G$).

The goal of this paper is to specify the set of all solutions of these minimization problems. We prove the following statements.

Theorem 1 *It holds:*

$$\min_{G \text{ has } < n \text{ corners}} s(G) = 1. \tag{1}$$

Remark 1

(*i*) *The equality (1) is attained for some continual set* $\mathbf{N}_n$ *of approximating stepfunctions;*

(*ii*) *All optimal stepfunctions* $G \in \mathbf{N}_n$ *have* $n-1$ *corners and all but maybe one corners coincide with the corners of* L_n.

(*iii*) *At most one corner of optimal stepfunction* $G \in \mathbf{N}_n$ *has a one nonintegral coordinate. In this case just one of the coordinates of the corner is nonintegral.*

Theorem 2 *There exists a unique approximating stepfunction G_n^* with $n-1$ corners such that*

$$r(G_n^*) = \min_{G \text{ has } < n \text{ corners}} r(G).$$

Remark 2

(*i*) *All rectangles between L_n and G_n^* have the same area, which is the least positive zero of the polynomial*

$$T_n = \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} t^k.$$

(*ii*) *For some $c > 0$*

$$\frac{1}{4} < r(G_n^*) \le \frac{1}{4} + cn^{-2}.$$

(*iii*) *The corners of L_n and G_n^* alternate.*

Theorems 1 and 2 give a complete solution of the approximation problems.

2 Stepfunctions minimizing the area

First we establish some properties of s-optimal stepfunctions. We need some notions. A unit square with the diagonal points $(i+1, n-i+1)$ and $(i, n-i)$ for some i, $1 \le i < n$, is called *adjacent* to L_n. We denote by F_n the union of closed squares adjacent to L_n and by $\hat{L}_n$ the stepfunction whose corners are $(i+1, n-i+1)$, $i = 1, 2, ..., n-1$. Denote by M_k the set of all non-increasing stepfunction with exactly k steps and put $M_{\hat{n}} = \cup_{k<n} M_k$.

Remark 3 *If we remove any unit step of L_n or add any unit square adjacent to L_n, we obtain a stepfunction with $n-1$ corners whose area differs from the area of L_n just on 1. Therefore, if G^* is an s-optimal approximating stepfunction, then*

$$s(G^*) \le 1. \tag{2}$$

Lemma 1 *Let $G^* \in M_n$ be s-optimal. Then the following statements hold:*

(*i*) *G^* has exactly $n-1$ corners.*

(*ii*) *All corners of G^* belong to F_n.*

(*iii*) *If two corners of G^* are contained in some square adjacent to L_n, then they are corners of L_n.*

Proof.
We prove five properties from which statements (*i*)–(*iii*) follow.