Disconnected vertex sets and equidistant code pairs

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Abstract

Let A and B be two disjoint vertex sets of a graph such that there are no edges between A and B. We give an upperbound for $|A| \cdot |B|$ in terms of eigenvalues of matrix representations of the graph. Applied to graphs in the Hamming association scheme it leads to bounds for equidistant code pairs, which improve and reprove some known results.

1 Introduction

Throughout G is a graph with vertex set V. Two disjoint subsets A and B of V are *disconnected* if there is no edge between A and B. We define $\Phi(G)$ to be the maximum of $\sqrt{|A| \cdot |B|}$ for disconnected sets A and B in G. Suppose V is the set of words of length n over an alphabet $\{1, \dots, q\}$ and define two words adjacent if their Hamming distance (i.e. the number of coordinates in which they differ) is not equal to a fixed $\delta \in \{1, \dots, n\}$. Then a pair of disconnected sets becomes an *equidistant code pair*.

The quantity $\Phi(G)$ has an application in information theory and leads to a lower bound for the two-way communication complexity of functions defined on $V \times V$ that are constant over the non-edges of G. About ten years ago this application caused some activity in the study of equidistant code pairs. The best result is due to Ahlswede [1], who gives the exact value of $\Phi(G)$ for $q = 2$, 4 and 5, for every δ and n.

In this paper we will give a bound for $\Phi(G)$ in terms of the eigenvalues of a matrix associated with G. In case the complement of G is given by a relation of an association scheme the bound takes an easy form, which applied to the Hamming scheme leads to a bound for equidistant code pairs. This bound is not as accurate as Ahlswede's result, but it is more general and it turns out to be sharp for some values of q, n and δ, and for $q \to \infty$ for any fixed n and δ.

2 Disconnected vertex sets

Let $V = \{1, \ldots, v\}$. We define $\mathcal{M}(G)$ to be the collection of symmetric $v \times v$ matrices M with all row and column sums equal to 1, such that $(M)_{i,j} = 0$ if i and j are distinct non-adjacent vertices of G. Let $\lambda_1(M), \ldots, \lambda_v(M)$ denote the eigenvalues of a matrix $M \in \mathcal{M}(G)$, such that $\lambda_1(M)$ has eigenvector $\underline{1}$ (the all-one vector), so $\lambda_1(M) = 1$. Put

$$\lambda(M) = \max_{i \neq 1} |\lambda_i(M)|.$$

Lemma 2.1 *If A and B are disconnected vertex sets of G and $M \in \mathcal{M}(G)$, then*

$$\frac{|A| \cdot |B|}{(v - |A|)(v - |B|)} \leq \lambda^2(M).$$

Proof. See [7] Theorem 2.1, or [11] Lemma 6.1. □

Theorem 2.2 *For any $M \in \mathcal{M}(G)$*

$$\Phi(G) \leq v \frac{\lambda(M)}{1 + \lambda(M)} \ .$$

Proof. Put $\Phi = \Phi(G)$ and take A and B such that $\Phi^2 = |A| \cdot |B|$. Then by Lemma 2.1

$$\frac{\Phi^2}{\lambda^2(M)} \leq (v^2 - v(|A| + |B|) + \Phi^2) \leq (v^2 - 2v\sqrt{|A| \cdot |B|} + \Phi^2) = (v - \Phi)^2.$$

Clearly $v \geq \Phi$, so $\Phi \leq (v - \Phi)\lambda(M)$, which yields the required bound. □

In order to investigate when the bound of Theorem 2.2 is best possible, we define

$$\phi(G) = \min_{M \in \mathcal{M}(G)} v \frac{\lambda(M)}{1 + \lambda(M)}$$

and we let $\mathcal{M}'(G)$ denote the set of matrices from $\mathcal{M}(G)$ for which the above minimum is attained. Thus Theorem 2.2 becomes $\Phi(G) \leq \phi(G)$. To determine $\phi(G)$ we need to find a matrix in $\mathcal{M}'(G)$. For that purpose the automorphisms of G can be helpful.