The Character Variety of Free Algebras and Free Groups

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Introduction

Let V a complex vector space of finite dimension d carrying a nondegenerate symmetric bilinear form which we denote by $<,>$. We use the standard notation $SO(<,>)$ for the special orthogonal group of this form. A function

$$f : \underbrace{V \oplus V \oplus ... \oplus V}_{n} \longrightarrow \mathbf{C}$$

which is polynomial (in terms of coefficients once a basis of V is fixed) is called an invariant if

$$f(\sigma v_1, \sigma v_2, ..., \sigma v_n) = f(v_1, v_2, ..., v_n)$$

for $v_i \in V$ arbitrary and $\sigma \in SO(<,>)$. The invariants form a commutative $\mathbf{C}-$algebra. Examples are, for $n = 2$, the bilinear form itself:

$$V \oplus V \ni (v_1, v_2) \longrightarrow < v_1, v_2 > \in \mathbf{C}$$

and, for $n = d$, the determinant:

$$\underbrace{V \oplus V \oplus ... \oplus V}_{d} \ni (v_1, v_2, ..., v_d) \longrightarrow det(v_1, v_2, ..., v_d) \in \mathbf{C}.$$

The first type of invariants and polynomials built up from them are called even invariants, the second type skew. We quote from Hermann Weyl's "Classical Groups" ([7]) that each $n-$variable invariant is a linear combination of monomials which are even or, if $n \geq d$, have the form $f(v_1, v_2, ..., v_n) \cdot det(v_{i_1}, v_{i_2}, ..., v_{i_d})$ where f is again even and $\{i_1, i_2, ..., i_d\}$ is any choice of indices from $\{1, 2, ..., n\}$. The multiplicative structure of the ring of invariants is reflected by the identity

(1)

$$det(v_1, v_2, ..., v_d) \cdot det(w_1, w_2, ..., w_d) = det(< v_i, w_j >)_{1 \leq i,j \leq d}$$

where the determinant on V is normalized by being 1 for some orthonormal basis of V.

It is natural to ask whether or not (1) is the only type of relations in the ring of invariants. Parts of this paper concern this question, in a slightly modified version, in case the dimension of V is three. The context is classical: Let $\mathbf{C}^{2\times 2}$ the complex 2×2 matrix algebra be acted on by $PGL(2,\mathbf{C}) \cong PSL(2,\mathbf{C})$ via conjugation. Then

$$\mathbf{C}^{2\times 2} \cong \mathbf{C} \oplus V$$

splits into two invariant components: the multiples of the unit matrix and the subspace of trace 0 matrices. The trace function

$$\mathbf{C}^{2\times 2} \ni A \longrightarrow tr(A) \in \mathbf{C}$$

may be characterized by the requirements

(i)

$$tr is \mathbf{C} - linear,$$

(ii)

$$tr(ABC)+tr(CBA)-tr(A)tr(BC)-tr(B)tr(AC)-tr(C)tr(AB)+tr(A)tr(B)tr(C)=0$$

for all triples $A, B, C \in \mathbf{C}^{2\times 2}$, and

(iii)

$$tr(E) = 2$$

for the unit matrix E.

It may be used to define a nondegenerate quadratic form and a determinant on V:

(2)

$$< T_1, T_2 >= tr(T_1 T_2)$$

and

(3)

$$< T_1, T_2, T_3 >= tr(T_1 T_2 T_3).$$

Relation (1) then reads

(4)

$$< T_1, T_2, T_3 > \cdot < T'_!, T'_2, T'_3 >= -\frac{1}{2} det(< T_i, T'_j >)_{1\leq i,j\leq 3}.$$