REPRESENTATION TYPE AND STABLE EQUIVALENCE OF MORITA TYPE FOR FINITE DIMENSIONAL ALGEBRAS

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In this note we show that two finite dimensional algebras have the same representation type if they are stably equivalent of Morita type.

Stable equivalences of Morita type were introduced for blocks of group algebras by Broué [2], see also [7]. The concept is motivated by a result of Rickard. In [9], he proved that any derived equivalence between finite dimensional self-injective algebras Λ and Γ induces a stable equivalence of a particulary nice form. More precisely, he constructs bimodules ${}_\Lambda B_\Gamma$ and ${}_\Gamma C_\Lambda$ such that the corresponding tensor functors induce mutually inverse equivalences between the stable categories $\underline{\operatorname{mod}}\,\Lambda$ and $\underline{\operatorname{mod}}\,\Gamma$ of finite dimensional modules for Λ and Γ. Stable equivalences of this form are said to be of Morita type. They occur frequently, for instance in the theory of blocks of group algebras. Other examples are obtained from two finite dimensional algebras Λ and Γ which are tilted from each other. The corresponding trivial extensions $T\Lambda$ and $T\Gamma$ are derived equivalent self-injective algebras [8], and therefore stably equivalent of Morita type. In fact, Assem and de la Peña proved in [1] that $T\Lambda$ and $T\Gamma$ have the same representation type; however their methods are completely different from the approach presented in this paper.

The representation type of a finite dimensional algebra Λ is traditionally defined using the concept of a continuous one-parameter family in the category $\operatorname{mod}\Lambda$ of finite dimensional Λ-modules. For instance, Λ is of tame representation type if for every $n \in \mathbb{N}$ only finitely many such families are needed to parametrize all indecomposable Λ-modules of dimension n. An alternative approach uses so-called generic modules. This was suggested by Crawley-Boevey and he established for tame algebras a correspondence between continuous one-parameter families and generic modules [3]. In [6], we used generic modules to show that a stable equivalence $\underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Gamma$ induces a bijection between continuous one-parameter families in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$. However, without any extra assumption on the stable equivalence it remains an open question how the dimensions of the modules in $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$ are related. In this paper we settle the problem for stable equivalences which are induced by an appropriate functor $\operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$. Stable equivalences of Morita type are of this form, and therefore we can prove that two finite dimesional algebras have the same representation type if they are stably equivalent of Morita type.

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We now begin with some notation and recall briefly the definitions which are needed in this paper. Let Λ be an associative ring with identity. Denote by $\operatorname{Mod}\Lambda$ the category of (right) Λ-modules and let $\operatorname{mod}\Lambda$ be the full subcategory of finitely presented Λ-modules. A sequence of morphisms $0 \to L \to M \to N \to 0$ in $\operatorname{Mod}\Lambda$ is *pure-exact* if the induced sequence $0 \to \operatorname{Hom}(X,L) \to \operatorname{Hom}(X,M) \to \operatorname{Hom}(X,N) \to 0$ is exact for every X in $\operatorname{mod}\Lambda$. In this case the morphism $L \to M$ is called a *pure monomorphism*, and L is *pure-injective* if every pure monomorphism $L \to M$ splits. We shall always assume that every finitely presented Λ-module is pure-injective. For example, every artin algebra has this property. We denote by

$\operatorname{Pinj}\Lambda$ the full subcategory of pure-injective Λ-modules and the *Ziegler spectrum* $\operatorname{Zsp}\Lambda$ is the set of isomorphism classes of indecomposable pure-injective Λ-modules. A Λ-module M is *endofinite* if the length $\ell_{\mathrm{end}}(M)$ of M, when regarded in the natural way as an $\operatorname{End}_\Lambda(M)^{\mathrm{op}}$-module, is finite. Note that every endofinite module is pure-injective. An endofinite module is called *generic* if it is indecomposable but not finitely presented [3]. Given a Λ-module X we denote by $\operatorname{Add} X$ the smallest full subcategory of $\operatorname{Mod}\Lambda$ which contains X and is closed under taking arbitrary coproducts and direct summands. The *stable category* $\underline{\operatorname{Mod}}_X\Lambda$ has the same objects as $\operatorname{Mod}\Lambda$ but $\underline{\operatorname{Hom}}_X(M,N)$ is the group of all Λ-morphisms $M \to N$ modulo the subgroup of those which factor through an object in $\operatorname{Add} X$. For example, one obtains the usual stable module category $\underline{\operatorname{Mod}}\,\Lambda$ by taking $X = \Lambda$. Analogously, the categories $\underline{\operatorname{Pinj}}_X\Lambda$ and $\underline{\operatorname{mod}}_X\Lambda$ are defined.

We shall need the following lemma.

Lemma. *Let X be endofinite. Then every Λ-module M has a decomposition $M = M_X \coprod X_M$ such that X_M is a maximal direct summand of M which belongs to* $\operatorname{Add} X$.

Proof. The union of a chain of pure submodules is again a pure submodule. Also, $\operatorname{Add} X$ is closed under taking direct limits and every module in $\operatorname{Add} X$ is pure-injective [5, Corollary 9.8]. Thus, by Zorn's lemma there is a maximal pure submodule X_M of M contained in $\operatorname{Add} X$ which is a direct summand. □

The next proposition collects the essential properties of a stable equivalence $\underline{\operatorname{mod}}_X\Lambda \to \underline{\operatorname{mod}}_Y\Gamma$ which is induced by some functor $\operatorname{Mod}\Lambda \to \operatorname{Mod}\Gamma$.

Proposition. *Let $f\colon \operatorname{Mod}\Lambda \to \operatorname{Mod}\Gamma$ be a functor commuting with direct limits and products. Suppose there are endofinite modules X in $\operatorname{mod}\Lambda$ and Y in $\operatorname{mod}\Gamma$ such that f induces an equivalence $\underline{\operatorname{mod}}_X\Lambda \to \underline{\operatorname{mod}}_Y\Gamma$. Then the following holds:*

(1) *f induces an equivalence $\underline{\operatorname{Pinj}}_X\Lambda \to \underline{\operatorname{Pinj}}_Y\Gamma$.*

(2) *f induces a bijection $\operatorname{Zsp}\Lambda \setminus \operatorname{Add} X \to \operatorname{Zsp}\Gamma \setminus \operatorname{Add} Y$, $M \mapsto M_f$, such that $fM = M_f \coprod Y_{fM}$ and Y_{fM} is a maximal direct summand of fM in $\operatorname{Add} Y$.*

(3) *There exists $c \in \mathbb{N}$ such that $\ell_{\mathrm{end}}(fM) \le c \cdot \ell_{\mathrm{end}}(M)$ for all M in $\operatorname{Mod}\Lambda$.*

(4) *If M is a generic Λ-module, then M_f is generic.*

Proof. (1) The functor f induces the following commutative diagram of functors:

$$\begin{array}{ccc} \operatorname{mod}\Lambda & \xrightarrow{f} & \operatorname{mod}\Gamma \\ \downarrow & & \downarrow \\ \underline{\operatorname{mod}}_X\Lambda & \xrightarrow{\sim} & \underline{\operatorname{mod}}_Y\Gamma \end{array}$$

The canonical functor $\operatorname{mod}\Lambda \to \underline{\operatorname{mod}}_X\Lambda$ extends uniquely to a functor $p_\Lambda\colon \operatorname{Mod}\Lambda \to \varinjlim \underline{\operatorname{mod}}_X\Lambda$ which commutes with direct limits. Here, $\varinjlim \underline{\operatorname{mod}}_X\Lambda$ denotes the category $\operatorname{Flat}((\underline{\operatorname{mod}}_X\Lambda)^{\mathrm{op}}, \mathrm{Ab})$ of flat functors $(\underline{\operatorname{mod}}_X\Lambda)^{\mathrm{op}} \to \mathrm{Ab}$ which is characterized by the following three properties [6]:

(i) $\varinjlim \underline{\operatorname{mod}}_X\Lambda$ contains, up to equivalence, $\underline{\operatorname{mod}}_X\Lambda$ as a full subcategory.

(ii) $\varinjlim \underline{\operatorname{mod}}_X\Lambda$ is an additive category with direct limits.

(iii) Every object in $\varinjlim \underline{\operatorname{mod}}_X\Lambda$ is a direct limit of objects in $\underline{\operatorname{mod}}_X\Lambda$.

We obtain therefore the following commutative diagram of functors which commute with direct limits:

$$\begin{array}{ccc} \operatorname{Mod}\Lambda & \xrightarrow{f} & \operatorname{Mod}\Gamma \\ \downarrow{\scriptstyle p_\Lambda} & & \downarrow{\scriptstyle p_\Gamma} \\ \varinjlim \underline{\operatorname{mod}}_X\Lambda & \xrightarrow{\sim} & \varinjlim \underline{\operatorname{mod}}_Y\Gamma \end{array}$$

It follows from Proposition 2.2 and Proposition 3.2 in [6] that p_Λ induces an equivalence $\underline{\operatorname{Pinj}}_X\Lambda \to \operatorname{Pinj}(\varinjlim \underline{\operatorname{mod}}_X\Lambda)$ where the notion of purity in $\varinjlim \underline{\operatorname{mod}}_X\Lambda$ is