Local to Global Structure in Twin Buildings

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1 Introduction

Twin buildings are a generalization of spherical buildings, and the present paper shows that the local structure of a twin building with no "tree residues" determines its global structure (subject to a mild restriction on four cases of finite residues). This is a generalization of the same result for spherical buildings given by J. Tits in [Ti74](Theorem 4.1.2). That theorem is crucial in the classification of spherical buildings, and the present paper is a step towards an eventual classification of twin buildings.

The sense in which twin buildings generalize spherical buildings is the following. If W denotes the Coxeter group of a building, then spherical buildings are those for which W is finite. In such cases there exists a unique element w_0 of longest length in W, and when the W-distance between two chambers is w_0 one says they are *opposite* (they are then antipodal to one another in the unique spherical apartment containing both). When W is infinite there is no element of longest length, hence no opposites. However in a twin building there is the idea of a chamber in one of the two buildings being opposite a chamber in the other. This feature is essential to our work.

We mentioned above that our twin buildings have no "tree residues". This means that if s_i and s_j are distinguished generators of the Coxeter group W, then the order m_{ij} of $s_i s_j$ is finite. Without that restriction each building would contain rank 2 residues which are trees, and we would lose control of the global structure because there are non-isomorphic twinnings of a given tree. Twin trees have special features of their own, and a start on the study of such objects is made in [RT94].

Main Theorems

As mentioned above one of the main results in [Ti74] asserts that a building of spherical type is determined by its local structure. The techniques leading to a proof of this result can be applied also to twin buildings whose diagrams have no entry equal to infinity. This fact motivated Conjecture 1 in [Ti90], which we state here in a slightly different version. For the notation the reader is referred to Sections 2 and 3.

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Conjecture: *Let* $(\Delta = (\Delta_+, \Delta_-, \delta_*)$, $\Delta' = (\Delta'_+, \Delta'_-, \delta'_*)$ *be thick twin buildings of type* M, *where* $m_{ij} \neq \infty$ *for any* $i, j \in I$. *Given two pairs of opposite chambers* $(c_+, c_-) \in \Delta_+ \times \Delta_-$, $(c'_+, c'_-) \in \Delta'_+ \times \Delta'_-$ *and an isometry from* $E_2(c_+) \cup \{c_-\}$ *onto* $E_2(c'_+) \cup \{c'_-\}$, *then there is a unique extension to an isometry from* Δ *onto* Δ'.

A step towards a proof of the conjecture above is made in [Ti90], where the following result is stated.

Theorem 1.1: *The assumptions being as in the conjecture above, each isometry from* $E_2(c_+) \cup \{c_-\}$ *onto* $E_2(c'_+) \cup \{c'_-\}$ *extends to an isometry from* $\Delta_+ \cup \{c_-\}$ *onto* $\Delta'_+ \cup \{c'_-\}$.

The theorem above is proved in [Ti90] under the additional hypothesis, that each panel contains a sufficiently large number of chambers. A slightly different approach [Ro] shows that this hypothesis can be dropped. See also the remarks in [Ti90] 5.7.

Theorem 1.1 says that the local structure determines Δ_+ and - with some additional arguments - also Δ_-; however, it is not clear at this point, whether the twinning δ_* is determined as well. It is not even clear whether the extension in Theorem 1.1 is unique (see 5.9 (a) in [Ti90]). In this paper we prove that the extension is indeed unique, if the set of chambers opposite to a given chamber is connected, which is our condition (co). This is then used to prove the conjecture when Δ satisfies (co).

All buildings considered in this paper are assumed to have no "tree residues", i.e. there is no entry equal to infinity in the diagram. Our results are the following:

Theorem 1.2: *If* Δ *satisfies (co), then the extension in Theorem 1.1 is unique.*

The proof will be given at the end of Section 4.

Theorem 1.3: *If* Δ *satisfies (co), then any isometry from* $\Delta_+ \cup \{c_-\}$ *onto* $\Delta'_+ \cup \{c'_-\}$ *extends uniquely to an isometry from* Δ *onto* Δ'.

Theorem 1.3 will be proved in Section 5.

Corollary 1.4: *If* Δ *satisfies (co), then the local structure determines the global structure.*

Our last result is a local criterion for the validity of condition (co), which is probably well known for spherical buildings. By assumption all rank 2 residues are spherical, and since (co) is satisfied for most rank 2 buildings (see below), the following theorem gives a useful criterion for Δ to satisfy (co).

Theorem 1.5: *If every rank 2 residue satisfies (co), then* Δ *satisfies (co).*

This will be proved in Section 6. As a consequence of Theorems 1.1,1.3 and 1.5 we obtain the following.

Corollary 1.6: *If every rank 2 residue satisfies (co), then the local structure* $E_2(c_+)$ *determines the global structure.*