Excellent algebraic groups I

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Abstract: The notion of an excellent quadratic form has been introduced by M. Knebusch [10, 11]. It has an obvious analogue for a semisimple linear algebraic group G over an arbitrary field k: the group G is excellent if, for any field extension K of k, the anisotropic kernel of the K-group $G \times_k K$ can be defined by a k-group. The aim of this paper is to investigate excellence properties of special linear and orthogonal groups.

Introduction

Not much is known about anisotropic semisimple linear algebraic groups over arbitrary fields. In order to reduce this gap of knowledge we study an important class of semisimple groups which we call *excellent* algebraic groups.

If k is any field, a semisimple k-group G is *excellent*, if, for any field extension K of k, there exists an algebraic group H over k such that $H \times_k K$ is the anisotropic kernel of the K-group $G \times_k K$.

A corresponding notion for quadratic forms was introduced by M. Knebusch [11, 7.14, p. 6]. A regular quadratic form q over k is *excellent* if, for any field extension K of k, there exists a form η over k such that $\eta \otimes_k K$ is the anisotropic kernel of the quadratic form $q_K := q \otimes_k K$.

The notion of excellence also has an obvious analogue for an Azumaya algebra A over k (cf. Def. 1.1 below). It is proved here (Th. 1.10 and Th. 1.6) that the following three statements are equivalent: i) *A is excellent.* ii) *The special linear group $G = \mathrm{SL}_1(A)$ is excellent.* iii) *The index of A is squarefree.* We like to emphasize that the squarefreeness of the index is a crucial condition for several algebraic properties of Azumaya algebras related to rationality questions discussed in the past and recently, cf. [23, 15, 19, 20, 13, 4].

Using the index reduction formula of Schofield and van den Bergh [17, Th. 1.3], we obtain in Cor. 1.7 that an Azumaya algebra is excellent if and only if its index and its exponent coincide over every field extension of k.

In section 2, the notion of excellence for the special orthogonal group of a regular quadratic form over a field k of characteristic $\neq 2$ is discussed. We prove in Th. 2.2, that the projective quadrics of two regular forms over k are isomorphic if and only if their corresponding special orthogonal groups are isomorphic as k-groups. This and the generic splitting theory [8] yield in Th. 2.5 a necessary and sufficient criterion for the special orthogonal group of a regular form q to be excellent. It shows that the class of quadratic forms which give rise to excellent groups is bigger than the class of excellent forms in the sense of Knebusch. There are non excellent quadratic forms which have excellent orthogonal groups. For example, the so called “good forms” as introduced by Fitzgerald [5] have an excellent special orthogonal group, but are not always excellent (cf. Ex. 2.7 below). On the other hand, if q is excellent, then G also is excellent. It would be interesting to get an overview about the class of quadratic forms described in Th. 2.5.

Knebusch [10] introduced a generic splitting tower $k = F_0 \subset F_1 \subset \ldots \subset F_h$ of a non-split quadratic form q over k and proved in [11, Th. 7.14], that an anisotropic quadratic form q over k is excellent if and only if there is a quadratic form η_i over k such that $\eta_i \otimes_k F_i$ is the anisotropic kernel of $q \otimes_k F_i$ for all $i = 1, \ldots, h$. We prove analogous statements for the special orthogonal group $G = \mathrm{SO}(q)$ and for the special linear group $G = \mathrm{SL}_1(A)$ in 2.4 and 1.9, 1.10 below.

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1. Azumaya algebras and excellent groups of type ${}^1\mathrm{A}_n$

An Azumaya algebra A over k is a finite dimensional central simple k-algebra, and, by Wedderburn's theorem, there is a unique integer $m \geq 1$ and a central division algebra D over k which is unique up to k-isomorphism such that $A \simeq \mathrm{M}_m(D)$, where $\mathrm{M}_m(D)$ denotes the full matrix ring of m by m matrices over D. The index of A is defined by $\operatorname{ind} A = \sqrt{\dim_k D}$, and the exponent $\exp A$ of A is the order of the equivalence class $[A]$ in the Brauer group $\mathrm{Br}(k)$. It is known that $\exp A$ divides $\operatorname{ind} A$ and that every prime divisor of $\operatorname{ind} A$ divides $\exp A$, cf. [16, 14.4, Prop. b, p. 261].

We write A^m for the m-fold tensor product $A \otimes_k \cdots \otimes_k A$ and set $A^0 = k$. Let A^{op} be the opposite algebra of A so that $[A^{\mathrm{op}}] = [A]^{-1}$ in $\mathrm{Br}(k)$. We write $A \sim A'$ if $[A] = [A']$ and set $K^* = K \setminus \{0\}$ for any field K.

1.1 Definition. An Azumaya algebra A over k is said to be *excellent* if for any field $K \supseteq k$, there is a k-algebra C such that $C \otimes_k K$ is the underlying central division K-algebra of $A \otimes_k K$.

If A is excellent and D is the underlying central division algebra of $A \otimes_k K$ then there is a canonical way to find a k-algebra C such that $D \simeq C \otimes_k K$ as follows from 1.5 and 1.6 below.

1.2 Lemma. *Let A be an Azumaya algebra over k, let K be a field extension of k, and let D be the underlying central division algebra of $A \otimes_k K$. Then the following two statements are equivalent.*

i) *There is a k-algebra C such that $C \otimes_k K \simeq D$.*
ii) *There is an Azumaya algebra B over k such that $B \otimes_k K$ is split and* $\operatorname{ind}(A \otimes_k K) = \operatorname{ind}(A \otimes_k B)$.

Proof. i) implies ii): Define $B := A^{\mathrm{op}} \otimes_k C$.
Then $B \otimes_k K \simeq A^{\mathrm{op}} \otimes_k C \otimes_k K \simeq (A^{\mathrm{op}} \otimes_k K) \otimes_K D \sim D^{\mathrm{op}} \otimes_K D$ is split over K, and $\operatorname{ind}(A \otimes_k B) = \operatorname{ind} C = \operatorname{ind} D = \operatorname{ind}(A \otimes_k K)$.
ii) implies i): Let C denote the underlying central division algebra of $A \otimes_k B$ over k. Then $\operatorname{ind} C = \operatorname{ind}(A \otimes_k B) = \operatorname{ind}(A \otimes_k K) = \operatorname{ind} D$. Since $B \otimes_k K$ is split, the three K-algebras $C \otimes_k K$, $A \otimes_k B \otimes_k K$ and $A \otimes_k K$ are equivalent in the Brauer group of K and represented by the central division algebra D, hence $C \otimes_k K \simeq D$. $\square$

Let F be the function field of the Severi-Brauer variety of an Azumaya algebra B over k. In [1, 9.3], Amitsur showed that the kernel of the canonical homomorphism $\mathrm{Br}(k) \to \mathrm{Br}(F)$ is cyclic generated by the class $[B]$. The following lemma is an easy consequence of this result and the index reduction formula of Schofield/van den Bergh [17, Th. 1.3].