A NEW ASYMPTOTIC EXPANSION AND ASYMPTOTICALLY BEST CONSTANTS IN LYAPUNOV'S THEOREM

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Abstract. A new asymptotic expansion is obtained in Lyapunov's Central Limit Theorem for distribution functions of centered and normed sums of independent random variables which are not necessary identically distributed. It is applied to determine the asymptotically best constants in the Berry–Esseen inequality, thus solving problems about their optimal values raised by Kolmogorov and Zolotarev.

1. Introduction

The main result of this paper is a new asymptotic expansion in the Central Limit Theorem (CLT) for distribution functions of sums of independent non-identically distributed random variables having finite moments of the third order (see Theorem 2.1). This result differs from previous results since we do not require any additional restrictions on the distributions of summands (such as, for instance, Cramér's condition (C), see Petrov (1975, 1987)). Theorem 2.1 is new even in the particular case of identically distributed summands, since the remainder term is estimated uniformly over the class of distributions of summands with fixed value of Lyapunov's fraction L. As an application of Theorem 2.1, we compute the asymptotically best constants in the Berry–Esseen inequality (see Theorem 2.2 and Corollary 2.3)

Denote by $\mathfrak{X}_3$ the set of all non-zero random variables (r.v.'s) with mean zero and finite absolute moments of order 3. We denote the set of distribution functions (d.f.'s) of these r.v.'s by $\mathfrak{F}_3$. In the case of symmetric r.v.'s we denote the corresponding sets by $\mathfrak{X}_{3s}$ and $\mathfrak{F}_{3s}$ respectively. Consider a finite sequence of independent r.v.'s $X_1,\dots,X_n$ of the class $\mathfrak{X}_3$ such that X_j have d.f.'s F_j, characteristic

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functions (c.f.'s) f_j, variances σ_j^2, third moments α_j, third absolute moments β_j respectively. Write

$$s=\sqrt{\sigma_1^2+\dots+\sigma_n^2},\quad B=\beta_1+\dots+\beta_n,\quad L=\frac{B}{s^3},\quad Z=\frac{X_1+X_2+\dots+X_n}{s}.$$

We denote the d.f. of Z by F, and the corresponding c.f. by f.

According to Lyapunov's theorem, the d.f. F converges to the standard normal d.f. Φ in the Kolmogorov metric

$$\rho(F,\Phi)=\sup_x\bigl|F(x)-\Phi(x)\bigr|$$

if Lyapunov's fraction $L\to 0$. Moreover (see Berry (1941) and Esseen (1942)), there exists an absolute constant c such that

$$\rho(F,\Phi)\leqslant cL. \tag{1.1}$$

Kolmogorov (1953) raised the question of finding the smallest value, say c_e, of the constant c in (1.1). He conjectured that $c_e=1/\sqrt{2\pi}$. Esseen (1956) showed that $c_e\geqslant(\sqrt{10}+3)/6\sqrt{2\pi}=1/\sqrt{2\pi}+0.0107899\dots$

Theorem A (Esseen (1956)). *Assume that $X_1,\dots,X_n$ are independent identically distributed (i.i.d.) and $F_1\in\mathfrak{F}_3$. Then*

$$\frac{\sigma_1^3}{\beta_1}\limsup_{n\to\infty}\sqrt{n}\,\rho(F,\Phi)\leqslant\frac{\sqrt{10}+3}{6\sqrt{2\pi}}.$$

This inequality turns into equality if and only if $F_1(x)$ is the lattice d.f., having jumps $(\sqrt{10}-2)/2$ and $(4-\sqrt{10})/2$ at the points $x_1=-h(4-\sqrt{10})/2$ and $x_2=h(\sqrt{10}-2)/2$ respectively, for some $h>0$.

Rogozin (1960) refined Esseen's result. Denote by $\mathcal{N}$ the set of all normal d.f.'s and $\rho(F,\mathcal{N})=\inf\limits_{Q\in\mathcal{N}}\rho(F,Q)$.

Theorem B (Rogozin (1960)). *Assume that $X_1,\dots,X_n$ are i.i.d. and $F_1\in\mathfrak{F}_3$. Then*

$$\frac{\sigma_1^3}{\beta_1}\limsup_{n\to\infty}\sqrt{n}\,\rho(F,\mathcal{N})\leqslant\frac{1}{\sqrt{2\pi}}.$$

This inequality turns into equality if $F_1(x)$ is the symmetric Bernoulli d.f., having jumps $\frac{1}{2}$ at $x=\pm1$.

The exact value of the constant c_e is not known until now. This problem is discussed in the monographs of Ibragimov and Linnik (1971), Petrov (1975, 1987)