PROJECTIVES ARE FREE FOR NILPOTENT VARIETIES

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Dedicated to the memory of Professor Tengiz Gegelia

0. Introduction. Projective objects in algebraic categories are retracts of free objects. We will show that finitely generated projective objects in the category of nilpotent groups of class n are free. The similar result is true for the category of nilpotent objects in other kind of algebraic categories, like Lie k-algebras, associative k-algebras etc. Here k is any commutative ring with property, that finitely generated projective k-modules are free. Moreover we will show that a retract of free finitely generated object is again free in complete case. We will derive such results from a very general statement on the lifting of idempotents in linear extensions of algebraic theories which generalize the well-known principle for modules (see for example Proposition III.2.10 of [1]).

1. Preliminaries on algebraic theories. We recall the notion of algebraic theory in the sense of Lawvere. Good reference for algebraic theories is [4]. The algebraic theory, or simply theory, formalizes the concept of an algebraic object as a set together with n-ary operations for various $n \in \mathbf{N}$ and equational relations.

1.1. Definition. *A theory is a category T with the following properties: T has a set of objects equal to the set of natural numbers $\mathbf{N} = \{0, 1, 2, \ldots\}$. The object n is the n-fold product of the object 1: there are morphisms $p_i^n \in \mathrm{Hom}_T(n, 1)$ $i = 1, \ldots, n$, such that the induced map*

$$\mathrm{Hom}_T(k, n) \ \rightarrow \ \mathrm{Hom}_T(k, 1)^n$$

is an isomorphism of sets. A morphism of theories is a product preserving functor which is the identity on objects. A theory T is called pointed, if 0 is a zero object in T.

One should think of the set $\mathrm{Hom}_T(n, 1)$ as the space of n-ary operations in the theory T.

1.2. Definition. *If T is a theory, a T-algebra X is a product preserving functor $X : T \to Sets$ to the category of sets. $X(1)$ is called the underlying set of the T-algebra X. A morphism of T-algebras is a natural transformation of functors, it is called surjective, if it yields surjection on the underlying sets.*

For a T-algebra X, each morphism $\phi \in \mathrm{Hom}_T(n, 1)$ gives rise to a map $X(1)^n \cong X(n) \xrightarrow{X(\phi)} X(1)$. This justifies thinking of a T-algebra as the underlying set together with n-ary operations parametrized by the set $\mathrm{Hom}_T(n,1)$.

1.3. Free T-algebras. The forgetful functor $T\text{-}alg \to Sets, X \mapsto X(1)$ has a left adjoint, the free T-algebra functor $F^T : Sets \to T\text{-}alg$. We let $< n >_T$ (or simply $< n >$) denote the free T-algebra generated by $\{1,\cdots,n\}$. The category T^{op} is equivalent to the category of finitely generated free T-algebras.

1.4. Algebraic functors. A morphism of theories $\phi : T_1 \to T$ induces an adjoint functor pair analogous to scalar restrictions and extensions: if X is a T-algebra, i.e., a product preserving functor $X : T \to Sets$, then the composite $X\phi : T_1 \to Sets$ is again product preserving, so it defines an T_1-algebra denoted by $\phi^* X$. Note that the underlying set of $\phi^* X$ is the same as for X. ϕ^* has a left adjoint ϕ_*. ([4, Thm. 3.7.7]). We remark that the functor ϕ_* sends free objects to free ones, and its restriction to finitely generated free objects is the dual of $\phi : T_1 \to T$.

1.5. Examples. i) Let R be a ring and R^{mod} be the category, whose objects are the natural numbers, while the morphisms are given by

$$\mathrm{Hom}_{R^{\mathrm{mod}}}(n,m) := \mathrm{Hom}_R(R^m, R^n).$$

Here Hom_R denotes the Hom in the category of left R-modules. Then R^{mod} is a theory called the theory of left R-modules. One can proved, that R^{mod}-algebras are the same as the left R-modules. In this way we get a functor from the category of rings to the category of theories, which is full embedding. We see that theories are generalization of rings and we will try to generalize some well-known notion of rings to theories (see 1.6 and section 4).

ii) We recall that the lower cental series of a group G is defined by the induction:

$$\Gamma_1 G = G,\ \Gamma_{i+1}G \ = \ [G, \Gamma_i G],\ i \geq 0.$$

By definition G is a nilpotent group of class n if $\Gamma_{n+1}G = 0$. For any group G, the group $G/\Gamma_{n+1}G$ is a group of nilpotency class n. If G is a free group generated by the set $\{1,\cdots,m\}$, then $< m >_n := G/\Gamma_{n+1}G$ is called free group of nilpotency class n generated by the set $\{1,\cdots,m\}$. One can consider the full subcategory of the category of nilpotent groups of class n whose objects are $< m >_n,\ m \geq 0$. We let $\mathbf{Nil_n(Gr)}$ denote the dual of this category. Then $\mathbf{Nil_n(Gr)}$ is a theory, called theory of nilpotent groups of class n. Obviously the category of $\mathbf{Nil_n(Gr)}$-algebras is isomorphic to the category of nilpotent groups of class n. The functor $G \mapsto G/\Gamma_{n+1}G$ yields the morphism of algebraic theories:

$$-/\Gamma_{n+1} : \mathbf{Nil_{n+1}(Gr)} \to \mathbf{Nil_n(Gr)}.$$