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SUBMODULE STRUCTURE OF GENERALIZED VERMA MODULES INDUCED FROM GENERIC GELFAND-ZETLIN MODULES

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Abstract. For complex Lie algebra $sl(n,\mathbb{C})$ we study the submodule structure of generalized Verma modules induced from generic Gelfand-Zetlin modules over some subalgebra of type $sl(k,\mathbb{C})$. We obtain necessary and sufficient conditions for the existence of a submodule generalizing the BGG theorem for the Verma modules.

1. Introduction

One of the basic facts in the theory of highest weight representations of finite - dimensional complex Lie algebras, established by Bernstein - Gelfand - Gelfand ([1], [4]) is the description of the submodule structure of Verma modules in terms of the Weyl group action on the weight space, considered as a parameter space of the isomorphism classes of Verma modules. The original theorem asserts that

if $M(\lambda)$ and $M(\mu)$ are Verma modules with highest weights $\lambda-\rho$ and $\mu-\rho$, then $M(\mu)\subseteq M(\lambda)$ if and only if there exist reflections $w_{\beta_1}, w_{\beta_2},\dots,w_{\beta_r}$ in the Weyl group such that

$$\mu = w_{\beta_r}w_{\beta_{r-1}}\dots w_{\beta_1}(\lambda)\leq w_{\beta_{r-1}}\dots w_{\beta_1}(\lambda)\leq\dots\leq w_{\beta_1}(\lambda)\leq\lambda.$$

A generalization of these results in the case of generalized Verma modules (GVM), induced from the infinite-dimensional irreducible weight $sl(2,\mathbb{C})$-modules without highest and lowest weight, was obtained in the case of $sl(n,\mathbb{C})$ in [11] and in the case of arbitrary simple algebra except G_2 in [8]. Those articles deal with the so-called α-stratified GVM, i.e. induced from the irreducible weight $sl(2,\mathbb{C})$-module without highest and lowest weight. The criterion of the existence of a non-trivial homomorphism between α-stratified Generalized Verma modules is formulated (as in the classical case) in terms of the Weyl group action on some algebraic variety, parametrizing the set of corresponding GVM's.

We note the peculiar features of that description:

- the space of parameters $\mathcal{A}$ is a 2-sheets covering of the variety of the isomorphism classes of GVM's;
- the submodules are parametrized by the coset W/H, where H is the subgroup of W, corresponding to the root systems $\{\alpha, -\alpha\}$;
- the action of the Weyl group on $\mathcal{A}$ is not originated from the action on the weight space $\mathfrak{H}^*$ and does not induce an action on the isomorphisms classes of GVM.

In the present paper we try to develop the technique from [11], [8] for a new class of modules. We consider the standard inclusion $sl(k,\mathbb{C}) \hookrightarrow sl(n,\mathbb{C})$ on the first k rows and columns and investigate the submodule structure of the GVM's, associated with this inclusion. The simple $sl(k,\mathbb{C})$-modules V, from which we induce generalized Verma $sl(n,\mathbb{C})$-module $M(\lambda, V)$, are the so called generic Gelfand-Zetlin modules ([5]). It seems to be a natural generalization of the results disscused above. Especially, the answer is obtained in the same terms as in [1],[11],[8]. Moreover, our results coincide with the classical BGG-theorem in the case $k=1$ and with the results from [11],[8] in the case $k=2$.

Remark, that the modules appeared in our paper, are not "weight modules" in a standard sense. More precisely, in the case $k>2$ their weight spaces (with respect to Cartan subalgebra) are infinite-dimensional.

We follow closely the original proof from [1] because it does not require extra techniques (i.e. translation functors, Shapovalov determinant, Jantzen filtration etc.), that is not yet developed in a non-highest weight case.

For our goals we need a technique of Gelfand - Zetlin modules developed in [5]. This modules defined by some conditions of finite-dimensionality, but with respect to the so called Gelfand - Zetlin subalgebra, in spite of $S(\mathfrak{H})$ in classical case.

We define a special partial order on the new parameters of GVM and give our answer in terms of the Weyl group action on the set, parametrizing the family of GVMs. We also reformulate our results in terms of Bruhat order on the quotients S_n/S_k where S_k is the Weyl group corresponding to the subalgebra $sl(k,\mathbb{C})$, and give the criterion for GVM to be irreducible.

Let us briefly describe a structure of the paper. In section 2 we give some previous notations and results about Generalized Verma Modules. In section 3 we collect all the preliminaries about Gelfand-Zetlin modules, following closely [5] (see also [13]). The only difference is the way of parametrizing of these modules. Section 4 is dedicated to modules, induced from algebra of corank one. In the section 5 we introduce the set, parametrizing GVM and the action of the Weyl group on this set. Theorem 3 of this section is an analogue of Harish-Chandra isomorphism theorem ([4, Theorem 7.4.5]). In section 6 we define an analogue of the Kostant function in the relative situation and show that it describes the growth of GVM considered. It allows us to prove some analogue of [4, Theorem 7.6.6]. Finally, in section 7 we obtain the BGG-like criterion for the existence of a submodule in GVM, induced from the generic GZ-module. As a corollary we give a criterion for GVM to be irreducible and some results, related to the Bruhat order on the quotients of the Weyl group.

2. Generalized Verma Modules

Let $\mathbb{C}$ denotes the complex numbers, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{Z}_+$ denotes the set of all non-negative integers.

For a Lie algebra $\mathfrak{A}$ we will denote by $U=U(\mathfrak{A})$ the universal enveloping algebra of $\mathfrak{A}$, by $Z(\mathfrak{A})$ the center of U and by $\mathfrak{A}$-mod its category of finitely generated modules.