FLOWS ON HOMOGENEOUS SPACES AND DIOPHANTINE APPROXIMATION ON MANIFOLDS

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Abstract. We present a new approach to metric Diophantine approximation on manifolds based on the correspondence between approximation properties of numbers and orbit properties of certain flows on homogeneous spaces. This approach yields a new proof of a conjecture of Mahler, originally settled by V. G. Sprindžuk in 1964. We also prove several related hypotheses of Baker and Sprindžuk formulated in 1970s. The core of the proof is a theorem which generalizes and sharpens earlier results on non-divergence of unipotent flows on the space of lattices.

1. Introduction

We start by recalling several basic facts from the theory of simultaneous Diophantine approximation. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we let

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i, \quad \|\mathbf{x}\| = \max_{1 \le i \le n} |x_i|,$$

$$\Pi(\mathbf{x}) = \prod_{i=1}^{n} |x_i| \quad \text{and} \quad \Pi_+(\mathbf{x}) = \prod_{i=1}^{n} |x_i|_+,$$

where $|x|_+$ stands for $\max(|x|, 1)$. One says that a vector $\mathbf{y} \in \mathbb{R}^n$ is *very well approximable* (cf. [S2]), to be abbreviated as VWA, if the following two equivalent conditions are satisfied:

(i) for some $\varepsilon > 0$ there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that

$$|\mathbf{q} \cdot \mathbf{y} + p| \cdot \|\mathbf{q}\|^n \le \|\mathbf{q}\|^{-n\varepsilon} \tag{1.1}$$

for some $p \in \mathbb{Z}$;

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(ii) for some $\varepsilon > 0$ there are infinitely many $q \in \mathbb{Z}$ such that

$$\|q\mathbf{y} + \mathbf{p}\|^n \cdot |q| \le |q|^{-\varepsilon} \tag{1.2}$$

for some $\mathbf{p} \in \mathbb{Z}^n$.

The equivalence of (1.1) and (1.2) follows from Khintchine's Transference Principle, see e.g. [C, Chapter V]. It is well known (see [C, Chapter VII]) that almost every $\mathbf{y} \in \mathbb{R}^n$ is not VWA. A more difficult question arises if one considers almost all points $\mathbf{y}$ on a submanifold M of $\mathbb{R}^n$ (in the sense of the natural measure on M). In 1932 K. Mahler [Mah] conjectured that almost all points on the curve

$$M = \{(x, x^2, \dots, x^n) \mid x \in \mathbb{R}\} \tag{1.3}$$

are not very well approximable. V. Sprindžuk's proof of this conjecture (see [Sp1, Sp2]) has eventually led to the development of a new branch in approximation theory, usually referred to as "Diophantine approximation with dependent quantities" or "Diophantine approximation on manifolds". According to Sprindžuk's terminology, a submanifold $M \subset \mathbb{R}^n$ is called *extremal* if almost all $\mathbf{y} \in M$ are not VWA. Since 1964, extensive classes of extremal manifolds have been found by Sprindžuk and his followers, see [Sp3, Sp4] for a review. However, the following conjecture made by Sprindžuk in 1980 has remained unproved:

Conjecture $\mathbf{H_1}$ [Sp4]. *Let $f_1, \dots, f_n$ be real analytic functions in $\mathbf{x} \in U$, U a domain in $\mathbb{R}^d$, which together with 1 are linearly independent over $\mathbb{R}$. Then the manifold $M = \{\mathbf{f}(\mathbf{x}) \mid \mathbf{x} \in U\}$ is extremal.*

The case $n = 2$ was settled by Schmidt in 1964 [S1], and Schmidt's method was recently extended to the case $n = 3$ by V. Beresnevich and V. Bernik [BB].

Note that the validity of the above conjecture for polynomials of one variable follows from Sprindžuk's original proof of Mahler's Conjecture. However, there exists a strengthening of Conjecture H_1 which has not been proved even for the curve (1.3). Namely, one says that $\mathbf{y} \in \mathbb{R}^n$ is *very well multiplicatively approximable* (to be abbreviated as VWMA) if the following two equivalent conditions are satisfied:

(i) for some $\varepsilon > 0$ there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ such that

$$|\mathbf{q} \cdot \mathbf{y} + p| \cdot \Pi_+(\mathbf{q}) \le \Pi_+(\mathbf{q})^{-\varepsilon} \tag{1.1M}$$

for some $p \in \mathbb{Z}$;

(ii) for some $\varepsilon > 0$ there are infinitely many $q \in \mathbb{Z}$ such that

$$\Pi(q\mathbf{y} + \mathbf{p}) \cdot |q| \le |q|^{-\varepsilon} \tag{1.2M}$$

for some $\mathbf{p} \in \mathbb{Z}^n$.

In other words, instead of taking the nth power of the maximum of the "differences" $|qy_i + p_i|$ in (1.2) one simply multiplies them, and instead of taking the nth power of the maximum of the numbers $|q_i|$ in (1.1) one multiplies those of them which are different from zero. The equivalence of (1.1M) and (1.2M) can be shown by modification of the argument needed to prove the standard form of Khintchine's Transference Principle, see a remark in [Sp3, p. 69]. A manifold $M \subset \mathbb{R}^n$ is said to be *strongly extremal* if almost all $\mathbf{y} \in M$ are not VWMA. Clearly (1.i) implies (1.iM), $i = 1, 2$; therefore any strongly extremal manifold is extremal. In his book [B], A. Baker raised the question of confirming the strong extremality of the curve (1.3). Later, in his review [Sp4], Sprindžuk also stated