# INTERSECTION PAIRING IN HYPERBOLIC MANIFOLDS, VECTOR BUNDLES AND CHARACTERISTIC CLASSES

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November 10, 1996

Abstract. We present some constraints on the intersection pairing in hyperbolic manifolds. We also show that, given a closed negatively curved manifold $M$ of dimension at least three, only finitely many rank $k$ oriented vector bundles over $M$ can admit complete hyperbolic metrics.

## §0. Introduction

Hyperbolic manifolds are locally symmetric Riemannian manifolds of negative sectional curvature. Any complete hyperbolic manifold is a quotient of a rank one symmetric space by a discrete groups of isometries that acts freely. The main result is the following

**0.1. Theorem.** *Let $Y$ be a finite connected CW-complex such that $\pi_1(Y)$ is torsion free group which is not virtually nilpotent. Assume that $\pi_1(Y)$ has no nontrivial decomposition as an amalgamated product or an HNN extension over a virtually nilpotent group. Then, given a nonnegative integer $n$ and homology classes $[\alpha] \in H_m(Y)$ and $[\beta] \in H_{n-m}(Y)$, there exists $K > 0$ such that,*

*(1) for any continuous map $f : Y \to N$ of $Y$ into an oriented complete hyperbolic $n$-manifold $N$ that induces an isomorphism of fundamental groups, and*

*(2) for any embedding $f : Y \to N$ of $Y$ into an oriented complete hyperbolic $n$-manifold $N$ that induces a monomorphism of fundamental groups.*

*the intersection number of the cycles $f_*\alpha$ and $f_*\beta$ in $N$ satisfies $|\langle f_*\alpha, f_*\beta\rangle| \le K$.*

The assumptions on $\pi_1(Y)$ are used to invoke a compactness theorem due to Rips that the space of conjugacy classes of faithful discrete representations of $\pi_1(Y)$ into the group of isometries of a rank one symmetric space is compact. For example, these assumptions hold if $Y$ is a closed aspherical manifold of dimension at least three where $\pi_1(Y)$ is word-hyperbolic. In particular, we get the following

**0.2. Corrolary.** *Let $M$ be a closed oriented $K(\Gamma, 1)$ manifold of dimension $n \ge 3$ such that $\Gamma$ is word-hyperbolic. Then there is a finite subset $F \subset H^k(M)$ that contains the Euler class of any $\pi_1$-incompressible topological embedding of $M$ into a complete oriented hyperbolic $(n+k)$-manifold.*

*Thus, $F$ contains the Euler class of any oriented vector bundle $\mathbb{R}^k \to E \to M$ if $E$ admits a complete $\mathbb{F}$-hyperbolic structure.*

The corollary generalizes a result of Kapovich [Kap2] who proved that, given a closed oriented negatively curved manifold $M$ of dimension $n \ge 3$, there is a finite subset $F \subset H^n(M)$ that contains the Euler class of any rank $n$ vector bundle whose total space admits a complete *real* hyperbolic structure.

Kapovich [Kap2] also proves a similar result in case $M$ is a closed orientable surface. Namely, the Euler number of any oriented $\mathbb{R}^2$-bundle over a closed oriented surface of genus $g$ is bounded by $\exp(\exp(10^8(g-1)))$ provided the total space of the bundle is a complete real hyperbolic manifold.

1991 *Mathematics Subject Classification.* Primary: 53C15, 57R20, 22E40, 20H10;.

*Key words and phrases.* Hyperbolic manifold, vector bundle, intersection pairing, Euler class.

More generally, Kapovich [Kap2] proved that there is a universal function $C(-,-)$ such that for any incompressible singular surfaces $\Sigma_{g_1}$, $\Sigma_{g_2}$ in a complete oriented real hyperbolic 4-manifold we have $|\langle[\Sigma_{g_1}],[\Sigma_{g_2}]\rangle| \le C(g_1,g_2)$.

Reznikov [Rez] showed that for any singular surfaces $\Sigma_{g_1}$, $\Sigma_{g_2}$ in a closed oriented negatively curved 4-manifold $M$ with the sectional curvature pinched between $-k^2$ and $-K^2$ there is a bound $|\langle[\Sigma_{g_1}],[\Sigma_{g_2}]\rangle| \le C(g_1,g_2,k,K,\chi(M))$ for some universal function $C$.

Some vector bundles appear naturally as hyperbolic manifolds. For example, given a real hyperbolic manifold $M$ the manifold $M\times\mathbb{R}^k$ admits a real hyperbolic structure while the tangent bundle $TM$ admits a complex hyperbolic structure. However, aside from these trivial cases, it is not easy to construct hyperbolic structures on vector bundles.

All known examples are based on some subtle geometric constructions. An orientable $\mathbb{R}^2$-bundle over a closed oriented surface of genus $g$ is known to admit a real hyperbolic structure provided the Euler class $e$ of the bundle satisfies $|e|<g$ [Luo1,2] (cf. [GLT], [Kap1], [Kui1,2]). For nonorientable bundles over nonorientable surfaces the condition $|e|\le[\frac{g}{8}]$ implies the existance of a real hyperbolic structure [Bel1]. Complex hyperbolic structures exist on orientable $\mathbb{R}^2$-bundles over closed oriented surfaces provided $|e+2g-2|<g$ [GKL]. There are examples of real hyperbolic structures on nontrivial orientable $\mathbb{R}^2$-bundles over closed real hyperbolic 3-manifolds [Bel2].

For any stable characteristic class $c$ (e.g. the total Pontrjagin class) there is a uniform bound on the number of elements of the ring $H^*(M,\mathbb{Z})$ that can be represented as the class $c$ of the normal bundle of an immersion of $M$ into a $(G,X)$-manifold where $G$ is real algebraic or semisimple with finite center. In fact, the bound is the number of connected components of the space $\mathrm{Hom}(\pi_1(M),G)$.

Note that a vector bundle is determined up to a finite number of possibilities by its rational Pontrjagin classes and the rational Euler class. In particular, we get the following.

**0.3. Theorem.** *Let $M$ be a closed smooth manifold and $k$ be a positive integer. Assume that either $k$ is odd or $k>dim(M)$.*

*Then the set of isomorphism classes of the normal bundles of smooth immersions of $M$ into (possibly incomplete) hyperbolic $(n+k)$-manifolds is finite.*

Combining 0.2 and 0.3 we get

**0.4. Theorem.** *Let $M$ be a closed oriented negatively curved $n$-manifold with $n\ge 2$ and $k$ be a positive integer with $n+k\ge 5$. Then the set of isomorphism classes of the normal bundles of smooth $\pi_1$-incompressible embeddings of $M$ into a complete hyperbolic $(n+k)$-manifold is finite.*

*In particular, the total spaces of only finitely many smooth oriented rank $k$ vector bundles over $M$ can admit complete hyperbolic structures.*

Note that according to a result of Anderson [A] the total space of any vector bundle over a closed negatively curved manifold admits a complete metric with the sectional curvature pinched between two negative constants.

Finally, I would like to thank W. P. Goldman, M. Kapovich, J. M. Rosenberg, and C. Rourke for many helpful discussions.

## §1. Varieties of representations and vector bundles

Let $B$ be a finite connected $CW$-complex with the universal cover $\tilde{B}$. Let $G$ be a Lie group that acts effectively and analytically on a contractible $n$-manifold $X$. Given a homomorphism $\rho:\pi_1(B)\to G$, the group $\pi_1(B)$ acts on $\tilde{B}\times X$ via $\gamma(b,x)=(\gamma b,\rho(\gamma)x)$. The quotient map $(\tilde{B}\times X)/\pi_1(B)=E_\rho\to\tilde{B}/\pi_1(B)=B$ is a flat $(G,X)$-bundle with fiber $X$ and holonomy $\rho$. Isomorphism classes of flat $(G,X)$-bundles are in one-to-one correspondence with the space of conjugacy classes of representations $\mathrm{Hom}(\pi_1(B),G)/G$.

The vector bundle $\tilde{B}\times TX\to\tilde{B}\times X$ decsendes to a vector bundle $\nu$ over $E_\rho$. The bundle $\nu$ is "vertical" in the sense that its fibers are tangent to the fibers of $E_\rho$. Clearly, isomorphic $(G,X)$-bundles (disregarding the flat structure) have isomorphic vertical bundles.

Since $X$ is contractible, the bundle $E_\rho$ has a section $s$ which is unique up to homotopy. For homotopic sections $s_2$ and $s_1$, the bundles $s_1^*\nu$ and $s_2^*\nu$ are isomorphic, hence, the isomorphism class