Quaternion Generation of the 2-Torsion Part of the Brauer Group of a Local Quintic (Type of Splitting: (1,1,1,1,1))

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1 Introduction

Let X be a smooth projective connected variety over k of characteristic zero, and let $K = k(X)$ be its function field. Then $K|k$ is a function field, that is, K is a separable and finitely generated field extension of k. Let $\mathrm{Br}_{nr}(K|k)$ be the unramified Brauer group of K (with respect to k). It is known (see, e.g., [6]) that there is a natural inclusion of $\mathrm{Br}\ X$ in $\mathrm{Br}(K)$ (here $\mathrm{Br}\ X$ denotes the Brauer group of X [5]) and this inclusion identifies $\mathrm{Br}\ X$ with the unramified Brauer group $\mathrm{Br}_{nr}(K|k)$. (Below we shall write $\mathrm{Br}\ X$ instead of $\mathrm{Br}_{nr}(K|k)$, keeping in mind this identification).

The group $\mathrm{Br}\ X$ is of great importance for many problems (we mention the problems of rationality of varieties, the problem of Brauer-Manin obstructions to various Hasse principles and finally the classical problem of the description of finite-dimensional division algebras over function fields (see [6],[10], [11]).

In this paper we are interested in the problem of the description of $\mathrm{Br}\ X$ for smooth geometrically connected projective curves over local fields, more precisely, for finite extensions k of $\boldsymbol{Q}_p$. The classical class field theory gives us a complete description of the Brauer group $\mathrm{Br}(k)$ of k (and also of finite dimensional central division algebras over k). But if $k(X)$ is a function field of a smooth geometrically connected projective variety over k, the situation is much more complicated. In spite of many results obtained in connection with the developing of multidimensional class field theory ([8],[9]), unfortunately, we do not have a complete description as in the classical situation even in the case of curves X (a "complete" description here means a representation of each element of $\mathrm{Br}(k(X))$ by the corresponding central division algebra over $k(X)$, in other words, it means a description of all central division algebras over $k(X)$). The concept of local-to-global principles is to try to describe global division algebras in terms of local ones. In the case of curves X over k this requires to investigate the groups $\mathrm{Br}(k_v(X_v))$ for all valuations v of k, where k_v is the completion of k at v and $X_v = X \times_k k_v$. By Tsen's theorem, $\mathrm{Br}(k_v(X_v))$ is trivial

in case of a complex v, and its structure is well-known for real v according to results in [1], [2], [3] and [7].

Thus, in this situation, the case of non-Archimedean v is the main unknown case. In the following, by a local field we will understand a finite extension of the field of p-adic numbers $\boldsymbol{Q}_p$. Let now X be a smooth geometrically connected projective curve over a local field k. Then it follows from [4], that there exists an exact sequence

$$0 \longrightarrow \mathrm{Br}\ X \longrightarrow \mathrm{Br}(k(X)) \xrightarrow{\phi} \coprod_{P \in X} \chi(G_P) \xrightarrow{\psi} \chi(G) \longrightarrow 0.$$

Here G denotes the Galois group of a separable closure of k, and G_P denotes the Galois group of a separable closure of the residue class field of $P \in X$, while $\chi(G_P)$ and $\chi(G)$ are the groups of continuous characters of G_P and G respectively (for the definitions of ϕ and ψ see [4]). In view of this sequence the group Im ϕ is known, so the group Br X is of main interest for the description of $\mathrm{Br}(k(X))$. The structure of Br X is well-known in the case of curves of genus zero [4], but even in the case of elliptic curves the complete structure of Br X is unknown. More generally, one can consider a smooth connected projective model X of an affine curve defined by the equation $y^2 = f(x)$, where $f(x)$ is a polynomial with coefficients in k without multiple roots, and one can ask for the structure of Br X. One of the aims of our few papers is to answer some questions related to this. Since Br X is an Abelian periodic group, the main problem of its description is to describe its m-torsion part ${}_m\mathrm{Br}\ X$ as an abstract Abelian group for any m. Among those groups, ${}_2\mathrm{Br}\ X$ is of particular importance, because its elements are represented by central division algebras with involutions, and this group is, according to a result of Merkurjev, related to the theory of quadratic forms by the isomorphism ${}_2\mathrm{Br}(k(X)) \cong I^2/I^3$, where I is the fundamental ideal of the Witt ring of $k(X)$. Another important problem is to find an explicit representation of all elements of ${}_m\mathrm{Br}\ X$ (or at least for generators) by central division algebras.

Of course, solutions to these two problems strongly depend on the curve X. Both problems have trivial solutions in the case of curves of genus zero. For genus one curves one has more complicated situation. If X is an elliptic curve, both problems were solved completely in [13] for non-dyadic fields k and $m = 2$. There are also a few preliminary results in the case of dyadic fields and in the case of principal homogeneous spaces which will be published elsewhere. For hyperelliptic curves only the case of non-dyadic local curves with good reduction is considered (see [14]).

In this paper we begin a systematic study of the non-dyadic case of a quintic X without any restriction on its reduction. More precisely, our main object is the group ${}_2\mathrm{Br}\ X$ for X being a smooth projective model of the affine curve defined by the equation $y^2 = f(x)$, where $f(x)$ is a monic polynomial of fifth degree with coefficients in k without multiple roots. The representation of generators of ${}_2\mathrm{Br}\ X$ by quaternion algebras depends on the irreducible factors of $f(x)$ over k. We will consider all the possible decomposition cases of $f(x)$ step by step. In this paper we consider the case $(1, 1, 1, 1, 1)$, i.e., $f(x)$ is a product of linear factors. The main result is a complete list of quaternion algebras representing generators of the group ${}_2\mathrm{Br}\ X$. Furthermore, we give a negative answer (see part 3 of this sequence of papers on the type of splitting (1,2,2), remark 1, p. 2 and remark 2, p. 9), with respect to the following question: Does any central unramified $k(X)$-division algebra of exponent 2 have a maximal subfield which comes from some finite extension of k? Note that, in case where X is an arbitrary elliptic curve or a hyperelliptic curve with