# Decay rates of the inverse of nonsymmetric tridiagonal and band matrices

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**Abstract**

It is well known that the inverse $C = [c_{i,j}]$ of an irreducible nonsingular symmetric tridiagonal matrix is a Green matrix, i.e. it satisfies $c_{i,j} = u_i v_j$ for $i \leq j$ and two sequences of real numbers $\{u_i\}$ and $\{v_i\}$. A similar result holds for nonsymmetric matrices $A$. There the inverse is given by four sequences $\{u_i\}, \{v_i\}, \{y_i\}$, and $\{y_i\}$. Here we characterize certain properties of $A$ in terms of the $u_i, v_i, x_i$, and $y_i$ namely that $A$ is an $M$–matrix or positive definite. We also establish a relation of zero row sums and zero column sums of $A$ and pairwise constant $u_i, v_i, x_i$ and $y_i$. Moreover we consider decay rates for the entries of the inverse of tridiagonal and block tridiagonal (banded) matrices. For diagonally dominant matrices we show that the entries of the inverse strictly decay along a row or column. We give a sharp decay result for tridiagonal irreducible $M$–matrices and tridiagonal positive definite matrices. We also give a decay rate for arbitrary banded $M$–matrices.

## 1 Introduction

In many mathematical problems which gives rise to a linear system of equations the system matrix is tridiagonal or block tridiagonal. For example the numerical solution of partial differential equations mostly leads to tridiagonal (one dimensional problems) and block tridiagonal (higher dimensional problems). Therefore these classes of matrices have been extensively studied. A review of this topic is given by Meurant [Meu] for symmetric matrices. One of the most important results is established by Gantmacher and Krein [GK], who proved that the inverse of a symmetric irreducible tridiagonal matrix is a so-called Green matrix which is given by two sequences $\{u_i\}$ and $\{v_i\}$ of real numbers. A similar result is established by Ikebe. He proved that the inverse of a nonsymmetric irreducible tridiagonal matrix has a similar structure as a symmetric one and is given by four sequences $\{u_i\}$, $\{v_i\}$, $\{x_i\}$, and $\{y_i\}$. We call such a matrix a generalized Green matrix.

Considering (block) tridiagonal matrices and their inverses there are different kinds of problems. On one hand one wants to find explicit formulas for the inverses of (block) tridiagonal matrices and on the other hand one wants to characterize matrices whose inverses are (block) tridiagonal. Here we continue considering tridiagonal matrices and their inverses. We consider generalized Green matrices and show that the inverse of a generalized Green matrix is tridiagonal. Thus, with Ikebe's result we obtain a characterization of irreducible nonsingular tridiagonal matrices. Moreover we characterizes tridiagonal $M$–matrices and tridiagonal symmetric positive definite matrices in terms of the $\{u_i\}$, $\{v_i\}$, $\{x_i\}$, and $\{y_i\}$. We explain the connection of zero row sums and zero columns of a tridiagonal matrix and pairwise constant $u_i, v_i, x_i$ and $y_i$.

It has been observed that for large classes of banded matrices the entries of the inverse tend to zero as $|i-j|$ becomes larger. The rate of decay is important to construct sparse approximations of the inverse as preconditioners. In [D] and [DMS] it is shown that the entries of the inverse of a symmetric positive matrix are bounded in an exponentially decaying manner along a row or column. For nonsymmetric tridiagonal $M$–matrices which are strictly diagonally dominant by rows and by columns we establish that the entries of the inverse indeed decay along each row and column away from the diagonal. We also give a bound for this decay. This result generalize a result by Concus, Golub, and Meurant [CGM] for symmetric matrices.

For arbitrary tridiagonal $M$–matrices or positive definite matrices the entries of the inverse do not decay a long a row in general. However, here we establish a sharp decay rate related to two diagonal entries for the inverses of tridiagonal matrices. This result can be proved easily and continues the number of elegant results on tridiagonal matrices. For symmetric matrices the result is related to Vassilevski [Vas] result for which Cauchy-Bunyakowski-Schwarz constants are used. Therefore we establish a new proof of Vassilevski's result and obtain a new way to compute or estimate Cauchy-Bunyakowski-Schwarz constants for tridiagonal matrices. More-