On incidence algebras associated with regular cell decomposition of $\mathbb{S}^n$

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Abstract

We associate with each regular cell decomposition of $\mathbb{S}^n$ certain finite-dimensional incidence algebra. We obtain the global dimension of such algebras and prove that they are Koszul algebras. We also obtain a global dimension of some quotients of such algebras.

1 Introduction

The problem to construct a projective resolution for simple modules over finite-dimensional algebras is one of the central problem in ring theory. This question was investigated by several authors, see for example [9, 7, 11, 8, 16] and references therein.

In particular one of the central problems is to obtain minimal projective resolution which gives an information about a global dimension of an algebra.

Although it is known different methods for constructing a projective resolutions for simple modules over path algebras and their quotients ([1, 2]) there is no common methods for constructing a minimal resolutions (see also [14] for an injective resolution). There exists only some hard conditions under which it is known that classical resolution is minimal ([9]).

Moreover, usually answer is given in terms of some geometric realization of a path algebra ([10, 11, 14, 3]) which leads to rather complicated calculations in non-trivial cases.

In the subsequent paper we start our investigations in opposite direction. We define by some geometrical reason a class of incidence algebras. All those algebras are defined by the poset of cells of a regular cell decomposition of $\mathbb{S}^n$. It is easy to see that classical results lead to enormous number of difficult calculations to obtain resolutions of simple modules over such algebras. Nevertheless, we obtain minimal projective resolution for simple modules over such algebras in easy way using some special properties of regular cell decomposition.

Moreover, it happened that all our algebras belong to the class of Koszul algebras ([9, 4]) which play an important role in the study of quantuum groups ([12]).

Let us briefly describe a structure of the paper. In section 2 we collect all necessary preliminaries on incidence algebras. In section 3 we define our main object — an incidence algebra associated with regular cell decomposition of $\mathbb{S}^n$. In section 4 we construct projective resolution for simple modules over such algebras and prove that all our algebras are Koszul algebras. In section 5 we investigate a global dimension of some zero relation algebras those are quotient of our algebras. Finally in section 6 we illustrate our results by several examples.

Finally we have to note that all the results of this paper are inspired by [5], namely by famous BGG-resolution of simple finite-dimensional module over simple finite-dimensional complex Lie algebra.

2 Preliminaries

Let $\mathbb{F}$ denotes a fixed field of arbitrary characteristics.

Let P be a finite partially ordered set (poset) with a partial order $<$. In a natural way we assosiate with P its quiver (i.e. a finite oriented graph) Q as follows: the set Q_0 of vertices (points) of Q coincides with P and there is an arrow from x to y, $x, y \in Q_0$ if and only if $y < x$ and there is no $u \in Q_0$ such that $y < u < x$. Such a quiver is usually called oriented quiver ([6]). We will denote by Q_1 the set of arrows of the quiver Q.

A path of length m on Q is a sequence of arrows $\alpha_1, \dots, \alpha_m$ written $p = \alpha_m \alpha_{m-1} \dots \alpha_1$, such that the initial point of α_{i+1} coincedes with the terminal point of α_i for all $1 \le i \le m-1$. We also assume that to every $x \in Q_0$ corresponds a "trivial path" e_x with x as the initial and terminal point. We will denote by B the set of all pathes in Q.

Consider a path algebra $\mathbb{F}Q$ of a quiver Q. This is an $\mathbb{F}$-algebra with basis set B and a multiplication defined as follows:

1. $e_x^2 = e_x$ for any $x \in Q_0$;

2. $e_x \cdot p = p \cdot e_y = p$ if y is the initial point of p and x is the terminal point of p;

3. $p \cdot g = 0$ if the initial point of p does not coincide with the terminal point of g;

4. $p \cdot g = pg$ if the initial point of p coincides with the terminal point of g.

Let I be a two-sided ideal in $\mathbb{F}Q$ generated by all differences $p - g$ where $p, g \in B$; p and g have common initial and common terminal points.

The quotient algebra $\mathcal{A} = \mathbb{F}Q/I$ is called the incidence algebra of the poset P ([6]).

For $x, y \in P$ we will write $[x, y]$ for the following set: $\{z \in P \,|\, x < z < y\} \cup \{x, y\}$ in the case $x < y$ or $x = y$ and we put $[x, y] = \emptyset$ otherwise. We will also denote by $[\cdot, x]$ the set $\{z \in P \,|\, z < x\} \cup \{x\}$

In this paper by $\mathcal{A}$-module we mean finite-dimensional $\mathcal{A}$-module.

It is well-known (se for example [13]) that any module V over $\mathcal{A}$ can be considered as a set of $\mathbb{F}$-spaces V_x, $x \in Q_a$ and a set of linear operators $C(V, f)$, $f \in Q_1$ that satisfies all necessary relation caused by the factorization by I.

For an $\mathcal{A}$-module V we will denote by $\operatorname{Supp} V$ the set $\{x \in Q_0 \,|\, V_x \neq 0\}$.

Simple modules $S(x)$ over $\mathcal{A}$ are in $1-1$ correspondense with $x \in Q_0$ and have rather simple structure: $\operatorname{Supp} S(x) = \{x\}$ and $\dim(S(x))_x = 1$. Thus $C(S(x), f) = 0$ for any $e(x) \neq f \in B$ and $C(S(x), e_x)$ is the identity map.