Classification of simple Harish-Chandra modules over $\mathbb{Q}$ - Virasoro algebra

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Abstract

We give a complete classification of simple Harish-Chandra modules over $\mathbb{Q}$ - Virasoro algebra.

1 Introduction

A classification of the simple Harish-Chandra modules over complex Virasoro algebra $\mathcal{V}$ was given by O.Mathieu in remarkable paper [7]. According to this classification each simple Harish-Chandra $\mathcal{V}$-module is either highest (lowest) weight module or module from the so-called intermediate series.

There exist a lot of different generalizations of the classical Virasoro algebra and centerless Virasoro algebra (Witt algebra) studied by several authors (see for example [4, 5, 9, 10, 11]). Naturally there appears a question to give a classification of Harish-Chandra modules over all those algebras.

In the subsequent paper we consider the following generalization of the Virasoro algebra which is called $\mathbb{Q}$ - Virasoro algebra. We define $\mathfrak{G}$ to be a Lie algebra with the base $\{e(x) \mid x \in \mathbb{Q}\} \cup \{c\}$ and the Lie brackets given by

$$[e(x), e(y)] = (y-x)e(x+y) + \delta_{x,-y}\frac{x^3-x}{12}c; \quad x, y \in \mathbb{Q}$$

$$[c, e(x)] = 0, \quad x \in \mathbb{Q}$$

(here $\delta_{x,-y}$ is a Kronecker symbol). It is a natural generalization of the higher rank Virasoro algebras considered in [10].

An intermediate series for $\mathfrak{G}$ is a family of modules $V(a,b)$, $a, b \in \mathbb{C}$ defined as follows: $V(a,b)$ has the base v_x, $x \in \mathbb{Q}$ and the action of generating elements of $\mathfrak{G}$ is defined by

$$e(x)v(y) = (ax+y+b)v(x+y), \quad x, y \in \mathbb{Q}. \tag{1}$$

For $a \neq 0, 1$ or $b \notin \mathbb{Q}$ the module $V(a,b)$ is simple. In the opposite case it has two simple subquotients and one of them is a trivial $\mathfrak{G}$-module. We will also call the intermediate series the family of all simple subquotients of $V(a,b)$.

The main result of this paper states that:

Theorem 1. *Each simple Harish-Chandra module over $\mathfrak{G}$ is a module of intermediate series.*

We note that this result is slightly different from Mathieu theorem since it would be shown that non-trivial highest or lowest weight module over $\mathfrak{G}$ is not Harish-Chandra module.

Let us briefly describe the structure of the paper. In section 2 we collect all necessary preliminaries. In section 3 we describe the support of simple Harish-Chandra modules over $\mathfrak{G}$. This is an essential part of the proof of main theorem 1 that presented in section 4.

2 Preliminaries

Let $\mathbb{C}$ denotes field of complex numbers, $\mathbb{Q}$ denotes field of rational numbers, $\mathbb{Z}$ denotes ring of integers and $\mathbb{N}$ denotes set of positive integers. We will also denote by $\mathbb{Z}_+$ the set of non-negative integers and by $\mathbb{Q}_+$ the set of non-negative rationals.

For a Lie algebra $\mathfrak{A}$ we will denote by $U(\mathfrak{A})$ its universal enveloping algebra.

Consider a $\mathbb{Q}$ - Virasoro algebra $\mathfrak{G}$ and keep the notations from section 1. We denote by $\mathfrak{H}$ the Cartan subalgebra of $\mathfrak{G}$ generated by $e(0)$ and c and by $\mathfrak{H}_1$ the subalgebra generated by $e(0)$.

For a $\mathfrak{G}$-module V and $\lambda \in \mathfrak{H}^*$ set

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{H}\}.$$

A $\mathfrak{G}$-module V will be called a weight module provided

$$V = \bigoplus_{\lambda \in \mathfrak{H}^*} V_\lambda.$$

If $V_\lambda \neq 0$ we will call λ the weight of V and V_λ — the corresponding weight space. Since element c is central it acts by maltiplication on scalar on each weight indecomposable module. We will call the corresponding scalar the charge of V. Thus for indecomposable V we will also call the weight of V the corresponding element $\lambda \in \mathfrak{H}_1^*$. From now on weight will mean weight with respect to $e(0)$. For a weight $\mathfrak{G}$-module V we set $\operatorname{Supp} V$ to be the set of all non-zero weights of V. Finally a weight $\mathfrak{G}$-module V will be called Harish-Chandra module provided $\dim V_\lambda < \infty$ for all $\lambda \in \operatorname{Supp} V$. Since $\mathfrak{H}_1$ is one dimensional we identify it with $\mathbb{C}$ in standard way: we define $\lambda \in \mathbb{C}$ to be the eigenvalue of $e(0)$.

One can see that for any indecomposable weight $\mathfrak{G}$-module V and for any $\lambda \in \operatorname{Supp} V$ holds $\operatorname{Supp} V \subset \lambda + \mathbb{Q}$.

A $\mathfrak{G}$-module V will be called dense provided $\operatorname{Supp} V = \lambda + \mathbb{Q}$ for some $\lambda \in \mathfrak{H}_1^*$. A $\mathfrak{G}$-module V will be called pined provided $\operatorname{Supp} V = (\lambda + \mathbb{Q}) \setminus \lambda$ for some $\lambda \in \mathfrak{H}_1^*$. A $\mathfrak{G}$-module V will be caleed cutted provided $\operatorname{Supp} V \subset \lambda + \mathbb{Q}_+$ or $\operatorname{Supp} V \subset \lambda - \mathbb{Q}_+$ for some $\lambda \in \mathfrak{H}_1^*$.