COMPARISON OF a-FUNCTIONS

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ABSTRACT. We show agreement of Lusztig's a-function with the a-function in [3].

1. INTRODUCTION

In this note, W denotes a finite, simply-laced Weyl group, or the affine Weyl group $E_9 = \hat{E}_8$. Let Γ_g be the set of simple generators, which are parametrized by the nodes of the associated Coxeter graph Γ.

Every $w \in W$ may be written as a product $s_1 s_2 s_3 \cdots s_n$ of simple generators. If n is minimal, we call this product "reduced" and define $l(w) = n$ and $supp(w)$ to be the set of generators which appear in the product.

Let H be the Iwahori-Hecke algebra associated to W. This is an algebra over $\mathbb{Q}[q^{1/2}, q^{-1/2}]$ with generators T_s for each $s \in \Gamma_g$ satisfying the relations $T_s^2 = (q-1)T_s + q$, $T_s T_t = T_t T_s$ if $st = ts$, and $T_s T_t T_s = T_t T_s T_t$ if $sts = tst$, for distinct s, $t \in \Gamma_g$. Also, let $v = q^{1/2}$. This algebra has a basis T_w, $w \in W$, where we have $T_w = T_{s_1} \cdots T_{s_n}$ whenever $s_1 \cdots s_n$ is a reduced expression for w. There is a unique automorphism $^- : H \to H$ which sends v to v^{-1} and T_w to $T_{w^{-1}}^{-1}$.

In [4], Kazhdan and Lusztig construct a canonical basis c_w (denoted C'_w in [4]) of H characterized by the property that $c_w = \bar{c}_w = v^{-l(w)} \sum_{y \le w} P_{y,w}(q) T_y$, where $\le$ denotes the Bruhat order, and $P_{y,w}(q) \in \mathbb{Z}[q]$ is a polynomial of degree $\le \frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$ and $P_{w,w}(q) = 1$.

Define $\bar{H} = H / < c_{sts} >_{s,t \in \Gamma_g, sts=tst, s \neq t}$.

For any $s \in \Gamma_g$, let b_s be the projection of $c_s = v^{-1}(T_s + 1)$ in $\bar{H}$. In [2], it is shown that $\bar{H}$ is an algebra over $\mathbb{Q}[q^{1/2}, q^{-1/2}]$ generated by b_s with the relations: 1. $b_s^2 = (v + v^{-1}) b_s$, 2. $b_s b_t = b_t b_s$, if $st = ts$, and 3. $b_s b_t b_s = b_s$ if $sts = tst$, $s \neq t$.

Let W_c consist of those elements of W whose reduced expressions avoid substrings of the form sts, where s and t are non-commuting generators. For $w \in W_c$, one can define $b_w = b_{s_1} \cdots b_{s_n}$ where $w = s_1 \cdots s_n$ is any reduced expression for w. According to [1], the elements b_w, $w \in W_c$, form a basis of $\bar{H}$.

Associated with these two bases are left, right, and two-sided cell structures, and a certain "a-function". The purpose of this note is to compare these structures and, in particular, show that the a-functions agree on W_c.

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2. Preliminaries

2.1. **Cells.** See [4] for a definition of Kazhdan-Lusztig left, right, and two-sided cells. Let $a : W \to \mathbb{Z}$ denote the a-function in [5].

See [3] for a definition of monomial left, right, and two-sided cells. We denote by $\overset{m}{\leq}_L$, $\overset{m}{\leq}_R$, and $\overset{m}{\leq}_{LR}$ the left, right, and two-sided preorders as used in [3]. We denote by $\overset{m}{\sim}_L$, $\overset{m}{\sim}_R$, and $\overset{m}{\sim}_{LR}$ the notions of left, right, and two-sided equivalence as used in [3]. We denote by $L_m(w)$, $R_m(w)$, and $T_m(w)$ the left, right, and two-sided monomial cells which contain w. Let $a_m : W_c \to \mathbb{Z}$ denote the a-function in [3].

Let $x \in W$. Define $\mathcal{L}(x) = \{s \in \Gamma_g \mid l(sx) < l(x)\}$. Define $\mathcal{R}(x) = \mathcal{L}(x^{-1})$. Suppose there exists distinct s, $t \in \Gamma_g$ such that $sts = tst$ and $\{s,t\} \cap \mathcal{L}(x) = \{s\}$. Then exactly one of the elements sx or tx solves the equation $\#\{s,t\} \cap \mathcal{L}(w) = 1$ for w. Let ${}^*x$ denote this element. Similarly, define x^*.

Define an equivalence relation $\overset{*}{\sim}_L$ on W to be the transitive closure of the relation ${}^*x \overset{*}{\sim}_L x$. Similarly define $\overset{*}{\sim}_R$. Define $\overset{*}{\sim}_{LR}$ to be the equivalence relation which is obtained from the transitive closure of both $\overset{*}{\sim}_L$ and $\overset{*}{\sim}_R$. We call the equivalence classes of $\overset{*}{\sim}_L$, $\overset{*}{\sim}_R$, and $\overset{*}{\sim}_{LR}$ the left, right, and two-sided star equivalence classes.

2.2. **The set W_c.** We need the following property and its dual (see [1]).

Property 2.1. Let $w \in W_c$, $t \in \Gamma_g$. Assume $wt \notin W_c$. Then we may write $w = w_1 t s w_2$ reduced, t commutes with all $u \in supp(w_2)$, and $(st)^3 = 1$. Furthermore, if $w = w_1' t s' w_2'$ is another such expression, then $s = s'$.

Let P denote the set of subsets of Γ which consist of non-adjacent nodes, including the empty set, $\emptyset$. For any $S \in P$, let $i(S)$ be the product (in any order) of the simple generators corresponding to the nodes in S (with $i(\emptyset) = 1$).

Theorem 2.2. *Let $w \in W_c$ satisfy $w^2 = 1$. There exists a unique $S \in P$ and a unique $x \in W_c$ such that $w = x i(S) x^{-1}$ reduced. Furthermore, $a_m(w) = \#S$.*

Proof. See [3, propositions 3.1.1 and 5.3.1].

3. Kazhdan-Lusztig cells, monomial cells, and star equivalence.

Theorem 3.1. *Left, right, and two-sided monomial cells are equal, respectively, to left, right, and two-sided star equivalence classes.*

Proof. We require a number of lemmas.

Lemma 3.2. *If either x or x^* is in W_c, then both x and x^* are in W_c. If either x or ${}^*x$ is in W_c, then both x and ${}^*x$ are in W_c.*

Proof. By definition, we can assume, by switching x and x^* if necessary, that there exists s, $t \in \Gamma_g$ such that $s \in \mathcal{R}(x)$, $t \notin \mathcal{R}(x)$, $s \notin \mathcal{R}(x^*)$, and $x^* = xt$.