A NOTE ON THE $\mathbb{D}_n$-PATTERN

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Abstract. The $\mathbb{D}_n$-pattern is a class of tame vector space categories where no classification of the isomorphism classes of the indecomposable subspaces seems to be available. We show that the subspace category of a representative of this class is equivalent to the category of representations of a specific bush. For bushes the classification of the isomorphism classes of the indecomposable representations is known. Finally, we extend our results to some multiple vector space categories which are natural generalizations of those in the $\mathbb{D}_n$-pattern.

1. Vector space categories

We fix an algebraically closed field k. As general sources for notation we refer to [Ri2] and [GR]. In particular, by *aggregate* we denote a k-additive category with finite-dimensional morphism spaces such that idempotents split. By $\operatorname{ind} K$ we denote a *spectroid* of K that is a full subcategory which contains exactly one object out of each isomorphism class of indecomposable objects in K.

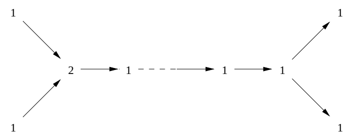
A *vector space category* is a pair (K, M) where K is an aggregate and $M : K \to \operatorname{mod} k$ is a k-linear functor. The *subspace category* $\check{\mathcal{U}}(K, M)$ of (K, M) is again an aggregate whose objects are the triples $X = (X_\omega, \gamma_X, X_0)$ such that X_ω is a finite-dimensional k-space, X_0 is an object of K and $\gamma_X : X_\omega \to M(X_0)$ is a k-linear map. Morphisms $X \to Y$ in $\check{\mathcal{U}}(K, M)$ are pairs $f = (f_\omega, f_0)$ consisting of a k-linear map $f_\omega : X_\omega \to Y_\omega$ and a morphism $f_0 : X_0 \to Y_0$ in K such that $M(f_0)\gamma_X = \gamma_Y f_\omega$. Sometimes one is only interested in the full subcategory $\mathcal{U}(K, M)$ of $\check{\mathcal{U}}(K, M)$ given by all objects X such that γ_X is a monomorphism. We call $\mathcal{U}(K, M)$ the *proper subspace category* of (K, M). The only object in $\operatorname{ind}\check{\mathcal{U}}(K, M)$ not belonging to $\operatorname{ind}\mathcal{U}(K, M)$ is $(k, 0, 0)$.

The vector space category (K, M) is called *faithful* if the functor M acts faithfully on K. We can always produce a faithful version (K_M, M) of a given vector space category (K, M) by defining K_M as $K/\operatorname{Ker} M$ where $\operatorname{Ker} M$ is the ideal of K given by all morphisms of K with $M(f) = 0$. The functor M acts faithfully on K_M in the natural way and there is an obvious full and dense functor $\check{\mathcal{U}}(K, M) \to \check{\mathcal{U}}(K_M, M)$. The indecomposable objects of $\check{\mathcal{U}}(K, M)$ which are mapped to 0 by this functor are precisely the ones which are isomorphic to $(0, 0, X_0)$ where X_0 is an indecomposable object of K satisfying $M(X_0) = 0$.

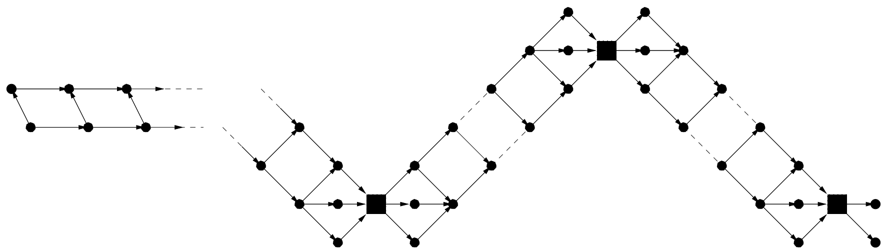
The reason for introducing vector space categories in the representation theory of finite-dimensional algebras is that there are various reduction functors which allow to reduce the study of the finite-dimensional modules over a finite-dimensional algebra to the study of the subspace category of a vector space category (see e.g. [Ri1], [Ri2], [G-V], [Dr1]). A prominent example of such a reduction technique appears for one-point extension algebras where the reduction functor is rather obvious. The one-point extension technique was used in [Ri1] in order to classify tame one-relation algebras. Therefore in this paper the tameness of several vector space categories had to be established. Some of these tameness proofs relied on further reductions such that only tameness could be obtained but the control on the indecomposable objects was lost.

An example of this effect are the vector space categories in the $\mathbb{D}_n$-pattern. *Patterns* are equivalence classes of vector space categories. The underlying equivalence relation was defined in [Ri1] in order to reduce the analysis of the subspace category to a single representative. Namely, two vector space categories (K, M) and (L, N) are said to be equivalent if there are cofinite (i.e. only finitely many indecomposable objects are not contained in the image) embedding functor $\alpha : K \to L$ and $\beta : L \to K$ such that $N\alpha \cong M$ and $M\beta \cong N$.

Canonical representatives of the $\mathbb{D}_n$-pattern, where $n \in \mathbb{N}$, $n \geq 4$, are the vector space categories (K_M, M) given as follows. The category K is the category $\operatorname{mod} H$ of finite-dimensional modules over the tame hereditary algebra $H = k\Delta$ where the underlying graph of the quiver Δ is an extended Dynkin diagram of type $\widetilde{\mathbb{D}}_n$. The functor M equals $\operatorname{Hom}_H(R, -)$ where R is an indecomposable regular module in $\operatorname{mod} H$ which is of regular length 2 and lies in a non-homogeneous tube of rank $n-2$. We choose one representative $\mathcal{D}_n$ which is given by the following quiver where to the vertices we attach the coefficients of the dimension vector which determines R uniquely up to isomorphism. Note, that we consider R as a covariant representation of the quiver.



A faithful vector space category like (K_M, M) is usually presented by the quiver with relations of its spectroid $\operatorname{ind} K_M$ and the dimensions of the spaces $M(X)$ for each object X in $\operatorname{ind} K_M$. If the dimensions of the spaces $M(X)$ are bounded by 2, then one draws a square for X if $\dim_k M(X) = 2$ and a dot for X if $\dim_k M(X) = 1$. With these conventions the data of the vector space category $\mathcal{D}_n$ look as follows:



Since the quiver of $\operatorname{ind} K_M$ is the translation subquiver of the Auslander-Reiten quiver of H supported by the indecomposable modules X satisfying $M(X) \neq 0$, we have to impose the usual mesh relations. It follows from [Dr2] that these are all necessary relations and that $\mathcal{D}_n$ is uniquely determined by the picture. But observe that, since the quiver has two infinite components, one has to consider suitable finite predecessor closed pieces of the first and successor closed pieces of the second component, in order to make use of [Dr2].

2. Multiple vector space categories

A *multiple vector space category* is also a pair (K, M) where K is an aggregate but $M : K \to \operatorname{mod} k^r$ is now a functor with values in the category of modules over some semisimple algebra k^r where $r \in \mathbb{N}$ is called the *number of peaks* (see [Si]). Thus vector space categories are the multiple vector space categories with one peak. The subspace category $\check{\mathcal{U}}(K, M)$ and also the proper subspace category $\mathcal{U}(K, M)$ are defined in the completely analogous way. Note that for an object $X = (X_\omega, \gamma_X, X_0)$ of $\check{\mathcal{U}}(K, M)$ the k^r-module X_ω can be considered as a tuple $(X_\omega^{(1)}, \ldots, X_\omega^{(r)})$ of finite-dimensional k-spaces. Analogously, the