STABLE EQUIVALENCE AND REPRESENTATION TYPE

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In this note we discuss various finiteness conditions which are preserved under a stable equivalence $\underline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Gamma$ between two artin algebras. More precisely, we prove:

- Λ has strongly unbounded representation type if and only if Γ has strongly unbounded representation type.
- Λ has no continuous pure-injective modules if and only if Γ has no continuous pure-injective modules.
- The Krull-Gabriel dimensions of Λ and Γ coincide.

We shall apply the machinery developed in [12]. There it was shown that Λ is (generically) wild if and only if Γ is (generically) wild.

1. Basic properties of stable equivalences

In this section we collect some basic properties of stable equivalences. Using generic modules, we prove at the end that a stable equivalence preserves strongly unbounded representation type. We begin with some notation and recall briefly the definitions which are needed in this paper.

Let Λ be an artin algebra. Denote by $\mathrm{Mod}\,\Lambda$ the category of (right) Λ-modules and let $\mathrm{mod}\,\Lambda$ be the full subcategory of finitely presented Λ-modules. A sequence of morphisms $0 \to L \to M \to N \to 0$ in $\mathrm{Mod}\,\Lambda$ is *pure-exact* if the induced sequence $0 \to \mathrm{Hom}_\Lambda(X, L) \to \mathrm{Hom}_\Lambda(X, M) \to \mathrm{Hom}_\Lambda(X, N) \to 0$ is exact for every X in $\mathrm{mod}\,\Lambda$. In this case the morphism $L \to M$ is called a *pure monomorphism*, and L is *pure-injective* if every pure monomorphism $L \to M$ splits. For example, every finitely presented Λ-module is pure-injective. We denote by $\mathrm{Pinj}\,\Lambda$ the full subcategory of pure-injective Λ-modules and the *Ziegler spectrum* $\mathrm{Zsp}\,\Lambda$ is the set of isomorphism classes of indecomposable pure-injective Λ-modules. A Λ-module M is *endofinite* if the length of M, when regarded in the natural way as an $\mathrm{End}_\Lambda(M)^{\mathrm{op}}$-module, is finite. Note that every endofinite module is pure-injective. An endofinite module is called *generic* if it is indecomposable but not finitely presented [4]. Given a finitely presented Λ-module X we denote by $\mathrm{Add}\,X$ ($\mathrm{add}\,X$) the smallest full subcategory of $\mathrm{Mod}\,\Lambda$ which contains X and is closed under taking arbitrary (finite) coproducts and direct summands. The *stable category* $\underline{\mathrm{Mod}}_X\,\Lambda$ has the same objects as $\mathrm{Mod}\,\Lambda$ but $\underline{\mathrm{Hom}}_X(M, N)$ is the group of all Λ-morphisms $M \to N$ modulo the subgroup of those which factor through an object in $\mathrm{Add}\,X$. For example, one obtains the usual stable module category $\underline{\mathrm{Mod}}\,\Lambda$ by taking $X = \Lambda$. Analogously, the categories $\underline{\mathrm{Pinj}}_X\,\Lambda$ and $\underline{\mathrm{mod}}_X\,\Lambda$ are defined.

We shall need the following lemma.

Lemma 1.1. *Every Λ-module M has a decomposition $M = M_X \coprod X_M$ such that X_M is a maximal direct summand of M which belongs to $\mathrm{Add}\,X$.*

Proof. The union of a chain of pure submodules is again a pure submodule. Also, $\mathrm{Add}\,X$ is closed under taking direct limits and every module in $\mathrm{Add}\,X$ is pure-injective since X is endofinite [10, Corollary 9.8]. Thus, by Zorn's lemma there is a maximal pure submodule X_M of M contained in $\mathrm{Add}\,X$ which is a direct summand. $\square$

The next proposition collects some basic properties of a stable equivalence between two artin algebras.

Proposition 1.2. *An equivalence* $f\colon \underline{\mathrm{mod}}_X\,\Lambda \to \underline{\mathrm{mod}}_Y\,\Gamma$ *has the following properties:*

(1) f *extends to an equivalence* $f'\colon \underline{\mathrm{Pinj}}_X\,\Lambda \to \underline{\mathrm{Pinj}}_Y\,\Gamma$.

(2) f *induces a bijection* $\mathrm{Zsp}\,\Lambda \setminus \mathrm{add}\,X \to \mathrm{Zsp}\,\Gamma \setminus \mathrm{add}\,Y$, $M \mapsto M_f$, *such that* $f'M \simeq M_f$ *in* $\underline{\mathrm{Pinj}}_Y\,\Gamma$.

(3) $M \in \mathrm{Zsp}\,\Lambda \setminus \mathrm{add}\,X$ *is a generic module if and only if* M_f *is generic.*

Proof. (1) The canonical functor $\mathrm{mod}\,\Lambda \to \underline{\mathrm{mod}}_X\,\Lambda$ extends uniquely to a functor $p_\Lambda\colon \mathrm{Mod}\,\Lambda \to \varinjlim \underline{\mathrm{mod}}_X\,\Lambda$ which commutes with direct limits. Here, $\varinjlim \underline{\mathrm{mod}}_X\,\Lambda$ denotes the category $\mathrm{Flat}((\underline{\mathrm{mod}}_X\,\Lambda)^{\mathrm{op}}, \mathrm{Ab})$ of flat functors $(\underline{\mathrm{mod}}_X\,\Lambda)^{\mathrm{op}} \to \mathrm{Ab}$ which is characterized by the following three properties [12]:

(i) $\varinjlim \underline{\mathrm{mod}}_X\,\Lambda$ contains, up to equivalence, $\underline{\mathrm{mod}}_X\,\Lambda$ as a full subcategory.

(ii) $\varinjlim \underline{\mathrm{mod}}_X\,\Lambda$ is an additive category with direct limits.

(iii) Every object in $\varinjlim \underline{\mathrm{mod}}_X\,\Lambda$ is a direct limit of objects in $\underline{\mathrm{mod}}_X\,\Lambda$.

It follows from Proposition 2.2 and Proposition 3.2 in [12] that p_Λ induces an equivalence $\underline{\mathrm{Pinj}}_X\,\Lambda \to \mathrm{Pinj}(\varinjlim \underline{\mathrm{mod}}_X\,\Lambda)$ where the notion of purity in $\varinjlim \underline{\mathrm{mod}}_X\,\Lambda$ is defined as for $\mathrm{Mod}\,\Lambda$. The assertion in (1) is now a consequence since f extends to an equivalence $\varinjlim \underline{\mathrm{mod}}_X\,\Lambda \to \varinjlim \underline{\mathrm{mod}}_Y\,\Gamma$.

(2) Every indecomposable pure-injective module has a local endomorphism ring. Therefore the canonical functor $\mathrm{Mod}\,\Lambda \to \underline{\mathrm{Mod}}_X\,\Lambda$ induces an injective map from $\mathrm{Zsp}\,\Lambda \setminus \mathrm{add}\,X$ into the set of isomorphism classes of indecomposable objects in $\underline{\mathrm{Pinj}}_X\,\Lambda$. The map is surjective by the preceding lemma. Using (1) we find for every $M \in \mathrm{Zsp}\,\Lambda \setminus \mathrm{add}\,X$ a module M_f such that $f'M$ and M_f are isomorphic in $\underline{\mathrm{Pinj}}_Y\,\Gamma$. It follows that the assignment $M \mapsto M_f$ gives a bijection between $\mathrm{Zsp}\,\Lambda \setminus \mathrm{add}\,X$ and $\mathrm{Zsp}\,\Gamma \setminus \mathrm{add}\,Y$.

(3) An object N in $\varinjlim \underline{\mathrm{mod}}_X\,\Lambda$ is said to be *endofinite* if $\mathrm{Hom}(A, N)$ is of finite length over $\mathrm{End}(N)^{\mathrm{op}}$ for all $A \in \underline{\mathrm{mod}}_X\,\Lambda$. Given a Λ-module M, it is easily checked that $p_\Lambda M$ is endofinite if M is endofinite. Also the converse is true: if $p_\Lambda M$ is endofinite, then M is endofinite [12, Corollary 3.3]. Moreover, $p_\Lambda M$ belongs to $\underline{\mathrm{mod}}_X\,\Lambda$ iff M is finitely presented. The assertion now follows from the construction of the bijection $M \mapsto M_f$. □

We now combine the preceding proposition with a result of Crawley-Boevey. Recall that an artin algebra Λ is said to be of *strongly unbounded representation type* if, for infinitely many $n \in \mathbb{N}$, there are infinitely many non-isomorphic indecomposable Λ-modules of length n.

Theorem 1.3. *Let* Λ *and* Γ *be artin* R*-algebras over a commutative local artinian ring* R *such that* $R/\mathrm{rad}\,R$ *is infinite. Suppose there exist an equivalence between* $\underline{\mathrm{mod}}_X\,\Lambda$ *and* $\underline{\mathrm{mod}}_Y\,\Gamma$ *for some modules* $X \in \mathrm{mod}\,\Lambda$ *and* $Y \in \mathrm{mod}\,\Gamma$. *Then* Λ *is of strongly unbounded representation type if and only if* Γ *is of strongly unbounded representation type.*

Proof. In [4, Theorem 7.3], Crawley-Boevey proved that an artin R-algebra is of strongly unbounded representation type iff there exists a generic Λ-module. The assertion therefore follows from Proposition 1.2. □

2. Continuous pure-injective modules

It is well-known that an artin algebra Λ is of finite representation type if and only if every pure-injective Λ-module is a coproduct of indecomposable modules [2]. In this section we exhibit the following weaker conditions: