Higher dimensional isoperimetric inequalities and uncombability of the Hilbert modular group

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Abstract

We prove the uncombability of the Hilbert modular group Γ , *corresponding to a totally real number field* K *of degree* ≥ 2.

Introduction

It has been an open problem which S-arithmetic groups and lattices in Lie groups are (bi)combable or (bi)automatic (for precise definitions see [ECHLPT]). Note the following diagram of inclusions between the classes $(\mathcal{B})\mathcal{C}, (\mathcal{B})\mathcal{A}$ of (bi)combable and (bi)automatic groups:

$$\begin{array}{ccc} \mathcal{C} & \supset & \mathcal{BC} \\ \cup & & \cup \\ \mathcal{A} & \supset & \mathcal{BA} \end{array}$$

It is not known whether or not any of the inclusions are equalities, "... although it is widely believed that there are combable groups which are not automatic"[Fa].

Thurston-Epstein have shown [ECHLPT] that $\mathrm{SL}_n(\mathbb{Z})$ is not combable for $n > 2$. (A purely algebraic proof of the fact that $\mathrm{SL}_n(\mathbb{Z})$ is not bicombable for $n > 2$ is contained in [BV]). Using a different method, Gersten-Short have shown [GS], [Sh] that $\mathrm{SL}_2(\mathcal{O})$ is not bicombable, where $\mathcal{O}$ is the ring of integers in the real quadratic number field $\mathbb{Q}(\sqrt{d})$. In [Pi] it is proven, that these groups are not combable.

Every lattice in $SO(n,1)$ is biautomatic. This follows from Epstein's result that geometrically finite hyperbolic groups are biautomatic [ECHLPT] and from the fact that lattices in $SO(n,1)$ are geometrically finite [Bo].

In [Fa] the complete solution of the bicombability problem is given in case of irreducible lattices in semisimple linear Lie groups. Namely, an irreducible lattice in a semisimple linear Lie group $G \neq SO(n,1)$ is bicombable if and only if it is cocompact.

The aim of this note is to prove the following

Theorem. *Let $\mathcal{O}$ be the ring of integers of the totally real field K of degree n over $\mathbb{Q}$. Then, for $n \geq 2$ the group $\mathrm{SL}_2(\mathcal{O})$ is not combable in the sense of* [ECHLPT]. *In particular it is not automatic* .

We combine approaches of [ECHLPT] and [Pi], where, resectively, the groups $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$ and $\mathrm{SL}_2(\mathcal{O})$, $\mathcal{O}$ real and quadratic, had been considered.

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