The repetitive algebra of a gentle algebra

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ABSTRACT. We are going to show that the repetitive algebra of a gentle algebra whose quiver has at least two cycles is non-domestic.

1. Introduction.

Let k be a field. Given a quiver Q and arrows $\alpha\colon a \to b, \beta\colon b \to c$, the concatenation of α and β will be denoted by $\alpha\beta$, it is a path of length 2 starting in a and ending in b. Recall that a relation for the quiver Q is a non-zero linear combination of paths of length at least 2 having the same starting point and the same end point.

Let Q be a quiver and ρ a set of relations for Q. The pair (Q, ρ) is said to be *special biserial* provided the following conditions are satisfied:

(1) Every vertex is starting point of at most two arrows.
(1′) Every vertex is end point of at most two arrows.
(2) For every arrow β, there is at most one arrow α with $\alpha\beta \notin \rho$.
(2′) For every arrow β, there is at most one arrow γ with $\beta\gamma \notin \rho$.

The pair (Q, ρ) is said to be *gentle* provided besides (1), (1′), (2), (2′) also the following conditions are satisfied:

(3) All the relations in ρ are monomials of length two.
(4) For every arrow β, there is at most one arrow α' with $\alpha'\beta \in \rho$.
(4′) For every arrow β, there is at most one arrow γ' with $\beta\gamma' \in \rho$.

A k-algebra A is said to be *special biserial*, or *gentle*, provided A is Morita equivalent to the factor algebra $kQ/\langle\rho\rangle$, where (Q, ρ) is special biserial, or gentle respectively; here kQ denotes the path algebra of the quiver Q, and $\langle\rho\rangle$ the ideal generated by the set ρ. (The k-algebras which we consider will always have sufficiently many idempotents, but not necessarily a unit element; note that the path algebra of a quiver Q has a unit element if and only if Q has finitely many vertices.)

Given a finite dimensional k-algebra A, Hughes and Waschbüsch [HW, see also H] have introduced the repetitive algebra $\widehat{A}$. It is easy to see (see Assem-Skowroński [AS] and Pogorzaly-Skowroński [PG]) that a finite dimensional algebra A is gentle if and only if the repetitive algebra $\widehat{A}$ is special biserial. The construction of $\widehat{A}$ for a gentle algebra A will be reviewed below.

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A special biserial k-algebra is always tame, and either domestic or of non-polynomial growth. Let A be a gentle algebra. It is known that in case the quiver $Q(A)$ of A has at most one cycle, then $\widehat{A}$ is domestic (see Nehring [N]). The aim of the present note is to show the converse:

Theorem. *Let A be a gentle algebra and assume that $Q(A)$ has at least two cycles. Then $\widehat{A}$ is non-domestic.*

It is known by the work of Nehring [N] that for a gentle algebra A with at least two cycles, the trivial extension $T(A)$ is non-domestic. The trivial extension $T(A)$ of A has $\widehat{A}$ as a Galois covering. Thus, our proof that $\widehat{A}$ is non-domestic also yields a proof that $T(A)$ is non-domestic.

The first part of the paper describes the relationship between the set $\mathcal{W}$ of words in the given quiver $Q(A)$ (here we consider all possible words, not taking into account the given set of relations), and a set $\widehat{\mathcal{W}}$ of words in the quiver of $\widehat{A}$ which do not involve certain paths. In fact, the set $\widehat{\mathcal{W}}$ is just the usual index set for the string modules of $\widehat{A}$ which are neither simple nor injective. We will construct a projection $\pi\colon \widehat{\mathcal{W}} \to \mathcal{W}$ and many sections σ^z and σ_z (with $z \in \mathbb{Z}$). In particular, we will see in which way words in $\mathcal{W}$ which are cyclic and of "cyclic defect zero" yield cyclic words in $\widehat{\mathcal{W}}$. All these considerations are well-known, but we hope that the formulations and proofs presented here will be found to be useful. In particular, we introduce in Section 4 as a tool the "expansion" of a locally bounded gentle quiver, this is again a gentle quiver, and $\widehat{A}$ is obtained from it by adding further relations. The essential part of the paper are Sections 8, 9 and 10, where we consider the case of a gentle quiver with at least two cycles: there, we are going to construct sufficiently many cyclic words belonging to $\mathcal{W}$ which have cyclic defect zero. Using the maps σ^z and σ_z, these words yield corresponding cyclic words in $\widehat{\mathcal{W}}$.

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2. Words in a quiver.

Let Q be a quiver. The concatenation of paths in Q will be written by juxtaposition (the concatenation w_1w_2 of paths w_1, w_2 is defined only in case the path w_2 starts in the same vertex as w_1 ends). If v, w are paths in Q, then v is called a *subpath* of w provided there are paths v_1, v_2 in Q such that $w = v_1vv_2$.

We denote by Q^* the opposite quiver of Q. Let $\overline{Q} = Q \sqcup Q^*$ be the quiver obtained from the disjoint union of Q and Q^* by identifying the corresponding vertices of Q and Q^*, thus $\overline{Q}$ is obtained from Q by adding to every arrow $\alpha\colon a \to b$ a new arrow $\alpha^{-1}\colon b \to a$, and we write $(\alpha^{-1})^{-1} = \alpha$. The arrows of $\overline{Q}$ will be considered as letters in order to form words. The original arrows of Q will be said to be the *direct* letters, those of Q^* will be called the *inverse* letters.

We denote by $\mathcal{W} = \mathcal{W}(Q)$ the set of paths of length at least 1 in $\overline{Q}$ which do not contain any subpath of the form $\alpha\alpha^{-1}$ where α is a letter, the elements of $\mathcal{W}$ will be called *words* (using $\overline{Q}_1$ as letter set). (Note that we are excluding the paths of length zero; in this way, we deviate from a convention which otherwise seems to be useful, but which would force us to take extra care: in the sequel, paths of length zero usually will not be needed or will behave differently.) If $w = l_1 \cdots l_n$ is a word with $l_i \in \overline{Q}_1$, then $w^{-1} = l_n^{-1} \cdots l_1^{-1}$ is called its *inverse.* If w belongs to $\mathcal{W}$, a subpath of w of length at least 1 will also be called a *subword.* If v, w are words,