# On Bicritical Snarks

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**Abstract**

Bicritical snarks are the irreducible ones with respect to the reductions considered by Nedela and Škoviera in [7]. We give a natural characterization of bicritical snarks and we characterize the elementary extensions of a snark. This solves Problem No. 6 of [7], and yields a method to construct all snarks starting from the irreducible ones.

Then we show that for some $n \geq 10$ and for each even $n \geq 92$ there is a bicritical snark of order $n$. This solves Problem No. 5 of [7].

## 1 Introduction

We are using standard graph theoretical terminology and notation in this paper. We define a *snark* to be a cubic graph with chromatic index $\chi' = 4$.
There are two main questions which lead to the study of reduction of snarks. The first one is the question about the intrinsic properties of cubic graphs which force them being a snark. The hope is that these properties can be detected in the irreducible snarks, and, since every snark is reducible to an irreducible one every snark should have this property, too.
The second one is the question about a method to construct all snarks recursively starting from the irreducible snarks. Here the hope is that the reverse operation (of a reduction) can be described without reflecting on the reduction process and hence it would yield such a method.
There are many papers dealing with this topic, cf. [1, 2, 5, 6, 7, 8, 9, 10, 11], and we refer the reader to one these papers for an more extensive introduction.
In this paper we consider the approach of Nedela and Škoviera [7]. They investigate the following reduction:
Let $G = (V(G), E(G))$ be a snark and let $F \subset E(G)$ be an $k$-edge cut ($k \geq 0$) whose removal divides $G$ into two components $H_1$ and $H_2$. If the chromatic index of one of the components is 4, say $\chi'(H_1) = 4$, then $H_1$ can be extended to a snark $H = (V(H), E(H))$

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with $|V(H)| \leq |V(G)|$, and $H$ is called a *$k$-reduction* of $G$. If $|V(H)| < |V(G)|$ then $H$ is called a *proper* $k$-reduction of $G$.
A snark is called *$k$-irreducible* if it has no proper $m$-reduction for each $m < k$, and it is called *irreducible* if it $k$-irreducible for each $k \geq 1$.
A snark $G$ is called *bicritical* if $\chi'(G - \{v, w\}) = 3$ for any two vertices $v, w \in V(G)$.

**Theorem 1.1** **([7])** *A snark is irreducible if and only if it is bicritical.*

Concerning the second question, i.e. to find a construction method for all snarks starting from the irreducible ones, Nedela and Škoviera introduce the notion of an elementary extension of a snark.
Let $G$ and $H$ be snarks. Then $G$ is called an *elementary extension* of $H$, if there is a set $F$ of two or three edges in $H$ such that $H - F$ can be extended to $G$ by adding two vertices and edges. Note that $\chi'(H - F) = 4$, since $H$ is a proper $k$-reduction of $G$, $k \in \{4, 6\}$.
Clearly, for each snark $G$ there is a sequence of snarks $G_1, G_2, \ldots, G_n$ such that $G_1$ is an irreducible snark, $G_n = G$, and $G_{i+1}$ is an elementary extension of $G_i$, $i = 1, \ldots, n-1$. The problem is to localize the set $F_i$ in $G_i$. Nedela and Škoviera formulated the following problem:

**Problem 1.2** **([7])** *For a given snark $G$, characterize all elementary extensions of $G$.*

In Section 2 we give a very natural characterization of the irreducible snarks which gives an answer to the first question from the beginning with respect to $k$-reductions. Then we solve Problem 1.2, which leads to a recursive construction method for all snarks starting from the irreducible ones.
Thus the two main questions in the field of snark reductions are answered for the approach of Nedela and Škoviera.
The set of the irreducible snarks is characterized by Theorems 1.1 and 2.1, but it is an infinite set. In [7] it is proved that the flower snark $J_{2k+1}$ is irreducible for each $k \geq 2$, where the flower snark $J_{2k+1}$ is the graph with vertex set $VJ_{2k+1} = \{a_i, b_i, c_i, d_i | i = 0, 1, \ldots, 2k\}$ and edge set $EJ_{2k+1} = \{b_i a_i, b_i c_i, b_i d_i; a_i a_{i+1}; c_i d_{i+1}; d_i c_{i+1} | i = 0, 1, \ldots, 2k\}$.
Thus there are irreducible snarks of $8k + 4$ for each $k \geq 2$. In [7] it is also stated that there are no irreducible of orders 12, 14, 16 and 24, but there are such snarks of orders 10, 18, and 22. In [1] these results are obtained independently, in fact there is shown that there are exactly 2 of order 18, 1 of order 20, 2 of order 22, 111 of order 26 and that there are 33 irreducible snarks of order 28.
Nedela and Škoviera asked the following questions:

**Problem 1.3** **([7])** *For which even number $n \geq 10$ does there exist an irreducible snark of order $n$? In particular, does there exist an irreducible snark of each sufficiently large order?*

In Section 3 we will give a partial solution of the first and a positive answer to the second question.