A category of pseudo-tangles with classifying space $\Omega^\infty S^\infty$

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1 Introduction

The category $\mathcal{F}_{Sym} = \coprod_{n \geq 0} \Sigma_n$ is the free symmetric monoidal category on one generating object, and it is well known that the group completion of its classifying space $B\mathcal{F}_{Sym}$ is $\Omega^\infty S^\infty$, i. e. stable homotopy of the 0-sphere.
If a monoidal category S satisfies certain properties, one obtains by Quillen's "S-inverse-construction" [1] a new category $S^{-1}S$ and a canonical arrow $S \longrightarrow S^{-1}S$ which yields a group completion $BS \longrightarrow B(S^{-1}S)$ on the level of classifying spaces.
$\mathcal{F}_{Sym}$ has a geometric interpretation: Morphisms $n \longrightarrow n$ are given by "braids with unconsidered crossings" (pseudo-braids) on n strings, composition by putting one picture on the top of the other and the tensor product by juxtaposition.
In this paper it is shown that $\mathcal{F}_{Sym}^{-1}\mathcal{F}_{Sym}$ is equivalent to a category $\mathcal{G}$ with an analogous geometric description and hence this category has classifying space $\Omega^\infty S^\infty$.
Section 2 of this article contains basic definitions, a short review of the S-inverse-construction and states the main theorem. Section 3 is devoted to the proof of the main theorem.

2 Definitions and theorems

2.1 Definition: Let $\mathcal{C}$ be a small category. Its nerve is the simplicial set whose k–simplices are diagrams in $\mathcal{C}$ of the form $X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_k$. The i-th face (resp. degeneracy) of this simplex is obtained by deleting the object X_i (resp. replacing X_i by $id : X_i \longrightarrow X_i$) in the evident way. The classifying space $B\mathcal{C}$ of $\mathcal{C}$ is the toplogical realization of its nerve .—

Some general facts about classifying spaces may be found in [4].

2.2 Convention: All terms concerning monoidal categories (sometimes also called tensor categories) are used in the sense of [2] .—

2.3 Remarks: The associativity isomorphisms of a small monoidal category describe a natural transformation from the functor $\otimes \circ (\otimes \times id)$ to the functor $\otimes \circ (id \times \otimes)$. Hence one obtains a homotopy associative multiplication on its classifying space which is in fact an A_∞-structure because of the pentagon rule. In case of a strict monoidal category the classifying space $B\mathcal{C}$ is a topological monoid.
This paper only deals with small strict monoidal categories, so the terms "small" and "strict" are omitted in the sequel.
If a monoidal category is even braided (cf. [2, p. 315]), the braid isomorphisms describe a natural transformation from the functor $\otimes$ to the functor $\otimes \circ T$, where $T : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}$ is the switch functor $(A, B) \longmapsto (B, A)$. Hence the classifying space of a braided monoidal category, in particular $B\mathcal{F}_{Sym}$, is a homotopy commutative monoid.—

2.4 Definition: An H-space X is called admissible if it is homotopy associative and if left translation by any given element is homotopic to right translation by the same element. X is called grouplike if $\pi_0 X$ is a group under the induced multiplication .—

Obviously the classifying space of a braided monoidal category is admissible.

2.5 Definition: Let $f : X \longrightarrow Y$ be an H-map between admissible H-spaces, where Y is grouplike. f is called a group completion if $f_* : H_*(X; R) \longrightarrow H_*(Y; R)$ is a localization of the ring $H_*(X; R)$ at its multiplicative submonoid $\pi_0 X$ for every commutative coefficient ring R .—

2.6 S^{-1}-Construction: Let S be a monoidal category. Define a category $S^{-1}S$ in the following way: $S^{-1}S$ has the same objects as $S \times S$, and a morphism from (A_-, A_+) to (B_-, B_+) is an isomorphism class of triples

$$(K, \;\; f_- : K \otimes A_- \longrightarrow B_-, \;\; f_+ : K \otimes A_+ \longrightarrow B_+)$$

where K is an object and f_-, f_+ are morphisms of S. An isomorphism between triples (K, f_-, f_+) and (K', f'_-, f'_+) is an isomorphism $K \xrightarrow{\cong} K'$ in S which makes the following diagrams commute: