Derived equivalence and stable equivalence of repetitions of algebras of finite global dimension

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Abstract. For repetitions (= repetitive "algebras") of algebras of finite global dimension we will show that there exists a triangle equivalence between their bounded derived categories of finitely presented modules iff there exists a triangle equivalence between their stable categories of finitely presented modules.

Introduction

Throughout this paper k denotes an algebraically closed field. The word algebra always means a finite-dimensional k-algebra with identity, and all categories and functors are assumed to be k-linear. Basic algebras are regarded as *spectroids* (= locally finite-dimensional categories) which are *finite* (=having only finitely many objects)([**2**]).

Consider the following sentences for *locally bounded spectroids* (= locally bounded categories) R and S:

- D(R, S) There is a triangle equivalence between the derived categories of the categories of modules over R and S (in this case R and S are called *derived equivalent*);
- Db(R, S) There is a triangle equivalence between the bounded derived categories of the categories of finitely presented modules over R and S (in this case R and S are called *derivatively equivalent* in [**2**, 12.5]); and
- St(R, S) (In the case that both R and S are selfinjective,) there is a triangle equivalence between the stable categories of the categories of finitely presented modules over R and S (in this case R and S are called *stably equivalent*).

Then first, by Rickard [**8**, Corollary 8.3] D(R, S) and Db(R, S) are equivalent if both R and S are finite. But if they have infinitely many objects, then this equivalence is not clear because Rickard's argument used in [**8**, Proposition 8.1] does not work in this case (see Example 2.6).

Second, assume that both R and S are selfinjective. Then by Keller-Vossieck [**6**, 2.3 Exemple] (or by the categorical version of Rickard [**9**, Theorem 2.1]) we see that Db(R, S) implies St(R, S). Sometimes the converse holds; e.g., in the case that R is a representation-finite selfinjective algebra of class A_n ([**1**, Theorem 5.8]).

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In this paper we show that the converse statement holds for repetitions (= repetitive "algebras") $A^{\mathbb{Z}}$ of algebras A of finite global dimension. Note that repetitions of (basic) algebras are selfinjective locally bounded spectroids and that locally bounded spectroids of this form are not very special (see Pogorzały [**7**, Theorem]).

Now let A and B be algebras. Then by Happel [**3**] or by Keller-Vossieck [**6**, 3.1] $\mathrm{St}(A^{\mathbb{Z}}, B^{\mathbb{Z}})$ implies $\mathrm{Db}(A, B)$ when both A and B are of finite global dimension. As explained above $\mathrm{Db}(A, B)$ and $\mathrm{D}(A, B)$ are equivalent. Further by [**1**, Theorem 1.5] $\mathrm{D}(A, B)$ implies $\mathrm{D}(A^{\mathbb{Z}}, B^{\mathbb{Z}})$. Hence if $\mathrm{D}(A^{\mathbb{Z}}, B^{\mathbb{Z}})$ implies $\mathrm{Db}(A^{\mathbb{Z}}, B^{\mathbb{Z}})$, then we are done. But as explained above this is not clear. In section 2 we then investigate when $\mathrm{D}(R, S)$ implies $\mathrm{Db}(R, S)$ for arbitrary spectroids R, S. We consider an intermediate condition $\mathrm{H}(R, S)$ that there is a triangle equivalence from the homotopy category of right bounded complexes of finitely generated projective modules over R to that over S. We show that $\mathrm{D}(R, S)$ implies $\mathrm{H}(R, S)$ if and only if the triangle equivalence in $\mathrm{D}(R, S)$ is *of right bounded type* (Lemma 2.3, Definition 2.4); and that if this is the case, then $\mathrm{D}(R, S)$ implies $\mathrm{Db}(R, S)$ (Lemma 2.5). In section 3 we show that the triangle equivalence in $\mathrm{D}(A^{\mathbb{Z}}, B^{\mathbb{Z}})$ constructed from that in $\mathrm{D}(A, B)$ as in [**1**] is of right bounded type (Proposition 3.1), which completes the proof of the assertion.

1. Preliminaries

For an additive category $\mathcal{A}$, we denote by $\mathcal{C}(\mathcal{A})$, $\mathcal{C}^{\mathrm{b}}(\mathcal{A})$ and $\mathcal{C}^{-}(\mathcal{A})$ the category of differential complexes, the category of bounded differential complexes and the category of right bounded differential complexes in $\mathcal{A}$, respectively; and by $\mathcal{H}(\mathcal{A})$, $\mathcal{H}^{\mathrm{b}}(\mathcal{A})$ and $\mathcal{H}^{-}(\mathcal{A})$ the corresponding homotopy categories. When $\mathcal{A}$ is an abelian category, we denote by $\mathcal{D}(\mathcal{A})$ and by $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ the corresponding derived categories. For each morphism f in $\mathcal{C}(\mathcal{A})$, $C(f)$ denotes the *mapping cone* of f; and $\overline{f}$ denotes the homotopy class of f in $\mathcal{H}(\mathcal{A})$.

Let R be a spectroid. We denote by $\mathrm{Mod}\, R$ the category of all (right) R-modules; by $\mathrm{mod}\, R$ the full subcategory of $\mathrm{Mod}\, R$ consisting of finitely presented objects; by $\mathrm{Pro}\, R$ the full subcategory of $\mathrm{Mod}\, R$ consisting of projective objects; by $\mathrm{pro}\, R$ the full subcategory of $\mathrm{Pro}\, R$ consisting of finitely generated projective objects; by $\mathcal{X}_R$ the full subcategory of $\mathrm{pro}\, R$ consisting of representable functors $R(-, x)$ for some $x \in R$; and by $\underline{\mathrm{mod}}\, R$ the (projectively) stable category of $\mathrm{mod}\, R$. We set $D := \mathrm{Hom}_k(-, k)$. By $\eta = \eta_R : R \to \mathcal{X}_R$ we denote the Yoneda isomorphism defined by $\eta(x) = R(-, x)$ for all $x \in R$. As in the notation $R(-, x)$, by (-) or (?) we denote a place for a variable. When we want to stress the difference of variables, we use different symbols like $R(-, ?)$.

Further $\mathcal{H}_{\mathrm{p}}(R)$ denotes the full subcategory of $\mathcal{H}(\mathrm{Mod}\, R)$ consisting of *homotopically projective* objects, i.e., formed by complexes P of R-modules such that $\mathcal{H}(\mathrm{Mod}\, R)(P, N) = 0$ for all acyclic complexes N of R-modules. (In Keller [**5**] homotopically projective complexes are called DG modules having property (P).) The inclusion $\mathcal{H}_{\mathrm{p}}(R) \to \mathcal{H}(\mathrm{Mod}\, R)$ admits the right S-adjoint ([**6**]) $\mathbf{p} : \mathcal{H}(\mathrm{Mod}\, R) \to \mathcal{H}_{\mathrm{p}}(R)$ and the canonical functor $\mathcal{H}(\mathrm{Mod}\, R) \to \mathcal{D}(\mathrm{Mod}\, R)$ induces a triangle equivalence $\mathcal{H}_{\mathrm{p}}(R) \to \mathcal{D}(\mathrm{Mod}\, R)$. By $\mathbf{p}$ we also denote a quasi-inverse of this equivalence. (See [**5**] for details.)

We recall some definitions needed later.

Definition 1.1. We say that spectroids A and B are *derived equivalent* if one of the following equivalent conditions holds ([**1**], [**5**], [**8**]):