DEGENERATIONS FOR MODULES OVER REPRESENTATION-FINITE BISERIAL ALGEBRAS*

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INTRODUCTION

Main results. Throughout the paper K denotes a fixed algebraically closed field. By an algebra A we mean an associative finite dimensional K-algebra with an identity, and by an A-module a finite-dimensional (unital) left A-module. We shall denote by $\operatorname{mod} A$ the category of A-modules, by Γ_A the Auslander-Reiten quiver of A, and by τ_A the Auslander-Reiten translation $D\mathrm{Tr}$ in Γ_A.

For an algebra A with basis $a_1 = 1, \dots, a_n$, we have the constant structures a_{ijk} defined by $a_i a_j = \sum a_{ijk} a_k$. The affine variety $\operatorname{mod}_A(d)$ of d-dimensional A-modules consists of n-tuples $m = (m_1, \dots, m_n)$ of $d \times d$-matrices with coefficients in K such that m_1 is the identity matrix and $m_i m_j = \sum m_k a_{ijk}$ holds for all indices i and j. The general linear group $\mathrm{Gl}_d(K)$ acts on $\operatorname{mod}_A(d)$ by conjugation, and the orbits correspond to the isomorphism classes of d-dimensional modules (see [16]). We shall agree to identify a d-dimensional A-module M with the point of $\operatorname{mod}_A(d)$ corresponding to it. We denote by $\mathcal{O}(M)$ the $\mathrm{Gl}_d(K)$-orbit of a module M in $\operatorname{mod}_A(d)$. Then one says that a module N in $\operatorname{mod}_A(d)$ is a degeneration of a module M in $\operatorname{mod}_A(d)$ if N belongs to the Zariski closure $\overline{\mathcal{O}(M)}$ of $\mathcal{O}(M)$ in $\operatorname{mod}_A(d)$, and we denote this fact by $M \leq_{\mathrm{deg}} N$. Thus $\leq_{\mathrm{deg}}$ is a partial order on the set of isomorphism classes of A-modules of a given dimension. It is not clear how to characterize $\leq_{\mathrm{deg}}$ in terms of representation theory .

There has been a work by S. Abeasis and A. del Fra [1], K. Bongartz [8], [9], [10], Ch. Riedtmann [19], and A. Skowroński and the author [22], [24] connecting $\leq_{\mathrm{deg}}$ with other partial orders $\leq_{\mathrm{ext}}$ and $\leq$ on the isomorphism classes in $\operatorname{mod}_A(d)$. They are defined in terms of representation theory as follows:

- $M \leq_{\mathrm{ext}} N$: $\Leftrightarrow$ there are modules M_i, U_i, V_i and short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ in $\operatorname{mod} A$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, and $N = M_{s+1}$ for some natural number s.

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- $M \leq N$: $\Leftrightarrow$ $[X, M] \leq [X, N]$ holds for all modules X.

Here and later on we abbreviate $\dim_K \mathrm{Hom}_A(X, Y)$ by $[X, Y]$. Then for modules M and N in $\mathrm{mod}_A(d)$ the following implications hold:

$$M \leq_{\mathrm{ext}} N \Longrightarrow M \leq_{\mathrm{deg}} N \Longrightarrow M \leq N$$

(see [9], [19]). Unfortunately the reverse implications are not true in general, and it would be interesting to find out when they are. This is the case for the representations of Dynkin quivers and Kronecker modules [9]. It was shown recently in [10] that $\leq_{\mathrm{deg}}$ and $\leq$ coincide for representations of extended Dynkin quivers.

In the representation theory of algebras an important role is played by biserial algebras. Following K. Fuller [13], an algebra A is called biserial if the radical of any indecomposable nonuniserial projective, left or right, A-module is a sum of two uniserial submodules whose intersection is simple or zero. A distinguish class of biserial algebras is formed by the special biserial algebras, introduced in [21]. These are the algebras which are Morita equivalent to bound quiver algebras KQ/I, where (Q, I) satisfies the following conditions

(S1) The numbers of arrows starting, respectively ending, in any fixed vertex of Q are bounded by 2.

(S2) For any arrow α of Q there is at most one arrow β and at most one arrow γ such that $\beta\alpha$ and $\alpha\gamma$ are not in I.

(S3) There is an upper bound for the lengths of paths in Q, which do not belong to I.

Further, a string algebra is a special biserial algebra satisfying the additional condition that I is generated by paths ("monomials").

It has been proved by A. Skowroński and J. Waschbüsh in [21] that every representation-finite biserial algebra is special biserial. Moreover, Thiele showed in [23] that for any compact Riemann surface $\mathcal{S}$ with nonempty boundary, there is a representation-finite string algebra $\Lambda(\mathcal{S})$ such that the geometric realization of its Auslander-Reiten quiver is homeomorphic to $\mathcal{S}$.

We are interested in degenerations of modules over representation-finite biserial algebras and our first main result is as follows.

THEOREM 1. *Let A be a representation-finite biserial algebra and M, N d-dimensional A-modules. Then $M \leq_{\mathrm{deg}} N$ if and only if $M \leq N$.*

A path $X_0 \to X_1 \cdots \to X_k$ in the Auslander-Reiten quiver Γ_A is called *sectional* if $\tau X_{i+1} \neq X_{i-1}$ for all $0 < i < k$. Our second theorem gives a characterization of representation-finite biserial algebras A for which the orders $\leq_{\mathrm{ext}}$ and $\leq_{\mathrm{deg}}$ do not coincide on $\mathrm{mod}\, A$.