Growth of polynilpotent varieties of Lie algebras and fast increasing entire functions

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Abstract

A growth function $c_n(\mathrm{V})$ for varieties of Lie algebras is studied; where $c_n(\mathrm{V})$ is the dimension of a linear span of all multilinear words on n distinct letters in a free algebra $F(\mathrm{V}, X)$ of the variety V. To any nontrivial variety of Lie algebras V corresponds the complexity function $\mathcal{C}(\mathrm{V}, z)$, which is an entire function of a complex variable. In case of a polynilpotent variety of Lie algebras V estimates for complexity function are studied, in most cases it is of infinite order. A connection between a growth of fast increasing entire function and an asymptotic of its Taylor coefficients is studied. The main result is an asymptotic for a function $c_n(\mathrm{V})$ in case of the polynilpotent variety of Lie algebras V. Also we prove an analog of Regev's theorem on an upper bound for an arbitrary variety of Lie algebras. As a result we obtain a better bound than an estimate, obtained by the author earlier.

1 Introduction: growth of varieties of Lie algebras

Let H be a Lie algebra Lie over a field K. Let $F = F(X)$ be a free Lie algebra with a countable set of free generators $X = \{x_i | i \in \mathbb{N}\}$. Then $0 \neq f \in F$ is said an *identity* (or *identical relation*) in H, iff for any $a_i \in H$, $i \in \mathbb{N}$ and any homomorphism $\phi : F \to H$, $\phi(x_i) = a_i$, $i \in \mathbb{N}$ we have $\phi(f) = 0$; also in this case we say that H is a *PI-algebra.* Suppose that $I \subset F$, then a class of all Lie algebras V, satisfying identities I, is said a *variety.* A set of all identities $\mathrm{V}(X)$, which are true in a given variety V, is an ideal in $F(X)$, so we can consider $F(\mathrm{V}, X) = F(X)/\mathrm{V}(X)$, which is called *free algebra of the variety* V. It has a property that $\forall H \in \mathrm{V}$, $\forall a_i \in H$, $i \in \mathbb{N}$, $\exists ! \phi : F(\mathrm{V}, X) \to H : \phi(x_i) = a_i$, $i \in \mathbb{N}$. A variety is called nontrivial if it satisfies some nonzero identity. For the theory of varieties of Lie algebras see a monograph [1].

Let $P_n(\mathrm{V}) \subset F(\mathrm{V}, X)$ be a set of multilinear elements on letters $x_1, \dots, x_n$. Namely, $P_n(\mathrm{V})$ is a linear span of all monomials on $x_1, \dots, x_n$ (more exactly, their images in $F(\mathrm{V}, X)$) such that each

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letter x_i, $i = 1, \ldots, n$ enters monomials exactly one time. For a given variety V appears a function of *codimension growth*

$$c_n(\mathrm{V}) = c_n(F(\mathrm{V}, X), X) = \dim_K P_n(\mathrm{V}), \quad n \in \mathbb{N}.$$

It is evident that for other variables $x_{i_1}, \ldots, x_{i_n} \in X$ we will get the same number. A behaviour of the sequence $c_n(\mathrm{V})$ is an important characteristic of the variety V.

The same observations hold also for varieties of associative algebras. About the codimension growth of associative algebras the following fact is known.

Theorem 1.1 (Regev, [2], see also [1]) *Let* V *be a variety of associative algebras which satisfies some nontrivial identity of degree d. Then*

$$c_n(\mathrm{V}) \le C^n, \quad n \in \mathbb{N}; \qquad \textit{where} \quad C = (d-1)^2.$$

Recall the definition of a polynilpotent variety of Lie algebras $\mathrm{N}_{s_q} \cdots \mathrm{N}_{s_2}\mathrm{N}_{s_1}$, corresponding to the tuple $(s_q, \ldots, s_2, s_1)$. If $q = 1$, then one has a variety of nilpotent Lie algebras N_{s_1}, defined by an identity

$$T_1(x_1, \ldots, x_{s_1}) = [x_1, \ldots, x_{s_1}] \equiv 0.$$

(Lie brackets we consider to be leftnormed, that is $[a_1, \ldots, a_n] = [\ldots[a_1, a_2], \ldots, a_n]$). Suppose that we have constructed an identity $T_{q-1}(x_1, \ldots, x_t)$, which defines $\mathrm{N}_{s_{q-1}} \cdots \mathrm{N}_{s_1}$, then $\mathrm{N}_{s_q} \cdots \mathrm{N}_{s_1}$ is defined by an identity

$$T_q(x_{11}, \ldots, x_{ts_q}) = [T_{q-1}(x_{11}, \ldots, x_{t1}), T_{q-1}(x_{12}, \ldots, x_{t2}), \ldots, T_{q-1}(x_{1s_q}, \ldots, x_{ts_q})] \equiv 0,$$

where x_{ij}, $1 \le i \le t$, $1 \le j \le s_q$ are distinct elements from X. It is easy to see that $\mathrm{N}_{s_q} \cdots \mathrm{N}_{s_1}$ consists of all Lie algebras H, such that there exists a chain of ideals $0 = H_{q+1} \subset H_q \subset \ldots \subset H_1 = H$, $H_i/H_{i+1} \in \mathrm{N}_{s_i}$. If $s_q = \ldots = s_2 = s_1$ then one has a variety of solvable Lie algebras A^q.

A growth of Lie algebras not exceeding an exponential growth has been studied extensively, see review [3]. Unlike associative case, (see Theorem 1.1) a growth of a rather small variety AN_2 is *overexponential* [4] (i.e. it cannot be bounded by any exponent). Moreover, in some sense, this situation is typical for Lie algebras. So, we have a vast area of the overexponential growth for Lie algebras.

In [5] there was suggested a scale to measure the overexponential growth of Lie algebras. In terms of this scale a growth of polynilpotent varieties of Lie algebras was described (see Theorem 2.1). The goal of the present paper is to give more precise description of the growth of polynilpotent varieties (Theorem 2.2, see also Lemma 3.3). As a result appears more fine scale for the overexponential growth (Section 2). Our main instrument is a *complexity function* of the variety, we study estimates on its growth (Section 3). Of independent interest is an asymptotic for the complexity function for the polynilpotent Lie variety, obtained in Theorem 3.2. In Section 4 we study an asymptotics for coefficients of fast increasing entire function, this is an important step in a proof of Theorem 2.2. In Section 5 we directly prove the lower estimates for Theorem 2.2. In Section 6 we prove an analog of Theorem 1.1 for Lie algebras (Theorem 2.3).

2 More fine scale for growth of Lie PI-algebras and growth of polynilpotent varieties

In [5] appeared the following types of overexponential growths. Denote

$$\ln^{(1)} x = \ln x, \ \ln^{(s+1)} x = \ln(\ln^{(s)} x); \quad \exp^{(1)} x = \exp x, \ \exp^{(s+1)} x = \exp(\exp^{(s)} x), \quad s \in \mathbb{N}.$$