The Ziegler spectrum of a tame hereditary algebra

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ABSTRACT. Let A be a finite dimensional hereditary algebra of tame representation type. Let $\operatorname{Com} A$ be a complete set of indecomposable algebraically compact A-modules (one from each isomorphism class). We are going to give an explicit description of the closed subsets of $\operatorname{Com} A$ in the sense of Ziegler.

1. Introduction

Let k be a field, let A be a connected finite dimensional hereditary k-algebra of tame representation type. For a description of the finite dimensional hereditary k-algebras we refer to [DR2]. Let $\operatorname{Mod} A$ denote the category of all A-modules and $\operatorname{mod} A$ the full subcategory of A-modules of finite length. The global structure of the category of all A-modules is well-known, see [DR1] and [R1].

Let $\operatorname{Com} A$ be a set of indecomposable algebraically compact A-modules, one from each isomorphism class. Let $\operatorname{ind} A$ be the subset of elements of $\operatorname{Com} A$ of finite length (since any finite length module is algebraically compact, $\operatorname{ind} A$ is just a complete set of indecomposable A-modules of finite length).

If $\mathcal{H}$ is a class of maps in $\operatorname{mod} A$, let $\mathcal{I}(\mathcal{H})$ be the full subcategory of all A-modules I with the following property: For any map $h\colon M \to M'$ in $\mathcal{H}$ and any map $f\colon M \to I$, there is $f'\colon M' \to I$ with $f'h = f$. A full subcategory of $\operatorname{Mod} A$ is said to be *definable* provided it is of the form $\mathcal{I}(\mathcal{H})$ for some class $\mathcal{H}$ of maps in $\operatorname{mod} A$; a full subcategory of $\operatorname{Mod} A$ is definable if and only if it is closed under products, direct limits and pure submodules (see [CB], 2.3 and Lemma 1 of 2.1; we will use the Trondheim survey of Crawley-Boevey [CB] as a general reference).

The subsets of $\operatorname{Com} A$ of the form $\mathcal{I}(\mathcal{H}) \cap \operatorname{Com} A$ are said to be *closed* (or Ziegler closed). It is obvious that the intersections of closed sets are again closed, thus any subset $\mathcal{X}$ of $\operatorname{Com} A$ has a closure which we will denote by $\overline{\mathcal{X}}$ (it is the intersection of all closed sets containing $\mathcal{X}$). The set $\operatorname{Com} P$ together with its closed subsets is called the *Ziegler spectrum* of A.

The aim of this note is to provide a direct approach for determining all closed subsets of $\operatorname{Com} A$. The question has been investigated before by several mathematicians. The case $\widetilde{D}_4$ was considered by Baur [B]. Parts of the answer can be found in an unpublished preprint of Prest [P1] and in the books of Prest [P2] and Jensen-Lenzing [JL]. The recent work of Geisler [G] has solved the problem in the case of a quiver algebra; his (rather tedious) proof is based on the inductive procedure of Donovan-Freislich [DF]. Prest [P3] has announced an independent approach

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dealing with the general case. The presentation given below was inspired by [G]; in order to reduce the computations presented there, we use the structure theory of the category of A-modules.

The elements of $\operatorname{Com} A$ are known (see for example [CB], Theorem 3.1). First of all, there are the elements of $\operatorname{ind} A$, they are either preprojective, regular, or preinjective. There are countably many preprojectives and countably many preinjectives in $\operatorname{ind} A$. There is a set Ω which parametrizes the socalled simple regular modules (these are modules which are not necessarily simple, but they are simple objects in the subcategory of all 'regular' modules — note that this is an abelian category, so that the notion of 'simplicity' is defined). If k is finite, Ω is countable, otherwise the cardinality of Ω is the same as the cardinality of k. For any simple regular module S and any $n \in \mathbb{N}_1$, there is a unique indecomposable module $S[n]$ which has a filtration of length n such that all the factors are simple regular and such that $S = S[1]$ occurs as a submodule of $S[n]$; in this way, one obtains all the regular modules in $\operatorname{ind} A$. For any simple regular module S, there is a chain of inclusions

$$S[1] \subset S[2] \subset S[3] \subset \cdots,$$

the union is indecomposable and is denoted by $S[\infty]$. Similarly, there is a chain of epimorphisms

$$S[1] \leftarrow S[2] \leftarrow S[3] \leftarrow \cdots,$$

the inverse limit is indecomposable and is denoted by $\widehat{S}$. A module of the form $S[\infty]$ is called a *Prüfer* module, a module of the form $\widehat{S}$ is called an *adic* module (the module $\widehat{S}$ itself is called the S-adic module, in the same way as one speaks of the p-adic integers). All the Prüfer modules and all the adic modules belong to $\operatorname{Com} P$. There is just one additional module in $\operatorname{Com} P$, the socalled *generic* module G. In the terminology of [R1], G is the unique indecomposable module which is torsionfree and divisible (a module X is said to be *torsionfree* provided that $\operatorname{Hom}(S, X) = 0$, and *divisible,* provided that $\operatorname{Hom}(X, S) = 0$, for all $S \in \Omega$).

Theorem. *A subset $\mathcal{X}$ of $\operatorname{Com} A$ is closed if and only if the following conditions are satisfied:*

(i) *If S is a simple regular A-module and if there are infinitely many finite length modules $X \in \mathcal{X}$ with $\operatorname{Hom}(S, X) \neq 0$, then $S[\infty]$ belongs to $\mathcal{X}$.*

(i*) *If S is a simple regular A-module and if there are infinitely many finite length modules $X \in \mathcal{X}$ with $\operatorname{Hom}(X, S) \neq 0$, then $\widehat{S}$ belongs to $\mathcal{X}$.*

(ii) *If there are infinitely many finite length modules in $\mathcal{X}$ or if there exists at least one module in $\mathcal{X}$ which is not of finite length, then the generic module belongs to $\mathcal{X}$.*

We see that the Ziegler closed subsets of $\operatorname{Com} A$ are related to the support of the functors $\operatorname{Hom}(S, -)$ and $\operatorname{Hom}(-, S)$, where S is simple regular. These functors have been considered before; in particular, we refer to the investigations of Geigle [Gg]. In [R2], the support $q(S)$ of the functor $\operatorname{Hom}(S, -)$ was studied in detail, in order to deal with one-point extensions. If one wants to combine similar cases, it was proposed in [R2] to consider equivalence classes called 'patterns', and some of these patterns have been exhibited there. All the informations needed in order to describe these patterns are contained in the tables of [DR1], this will be recalled in section 6.