ON THE MODALITY OF PARABOLIC SUBGROUPS OF LINEAR ALGEBRAIC GROUPS

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Abstract. For a linear algebraic group G over an algebraically closed field k and a parabolic subgroup P of G the *modality of* P is defined to be the maximal number of parameters upon which a family of P-orbits on $\operatorname{Lie} P_u$ depends and it is denoted by $\operatorname{mod} P$, where P_u is the unipotent radical of P. The principal aim of this note is a generalization of two basic "monotonicity" results from [19] to positive characteristic: (1) If Θ is a semisimple automorphism of G and P is Θ-stable, then $\operatorname{mod} P^\Theta \leq \operatorname{mod} P$. (2) If G is reductive, $\operatorname{char} k$ is a good prime for G, and H is a closed reductive subgroup of G normalized by a maximal torus $T \subset P$ of G, then $\operatorname{mod}(P \cap H) \leq \operatorname{mod} P$.

1. Introduction

Let G be a linear algebraic group over an algebraically closed field k and let P be a parabolic subgroup of G, i.e., G/P is a complete variety. Then P acts on its unipotent radical P_u via conjugation and on $\mathfrak{p}_u$, the Lie algebra of P_u, via the adjoint action. The modality of the action of P on $\mathfrak{p}_u$ is the maximal number of parameters upon which a family of P-orbits on $\mathfrak{p}_u$ depends and is denoted by $\operatorname{mod}(P : \mathfrak{p}_u)$; likewise for the action of P on P_u. The *modality of* P is defined as $\operatorname{mod} P := \operatorname{mod}(P : \mathfrak{p}_u)$. The principal aim of this paper is to generalize two "monotonicity" results on the modality from [19] to positive characteristic:

Theorem 1.1. *Let G be a linear algebraic group and let Θ be a semisimple automorphism of G. Assume that P is a Θ-stable parabolic subgroup of G. Then $\operatorname{mod} P^\Theta \leq \operatorname{mod} P$.*

Theorem 1.2. *Let G be a reductive algebraic group and suppose that* $\operatorname{char} k$ *is a good prime for G. Let H be a closed reductive subgroup of G normalized by a maximal torus T of G and P a parabolic subgroup of G containing T. Set $Q := P \cap H$. Then* $\operatorname{mod} Q \leq \operatorname{mod} P$.

Analogues of Theorems 1.1 and 1.2 were proved in [19, Theorem 2.13] under the assumption that $\operatorname{char} k$ is zero. The proofs in *loc. cit.* do not generalize to positive characteristic, as they make use of the separability of the orbit maps for the action of P on $\mathfrak{p}_u$.

These "monotonicity" theorems from [19] provide helpful inductive tools for comparing the modality of parabolic subgroups of various groups and, apart from [19] itself, have been used frequently in subsequent papers on the subject, such as [11], [18], [23], [24], and [25]. Theorems 1.1 and 1.2 allow us to extend the principal results from all of these papers to positive characteristic. See also [11] and [12].

These results are proved in Section 3. In the remaining parts of the paper we present some complements to the main theme. We briefly indicate the results.

In characteristic zero the exponential mapping is a P-equivariant isomorphism between $\mathfrak{p}_u$ and P_u, and thus $\operatorname{mod}(P : P_u) = \operatorname{mod}(P : \mathfrak{p}_u)$. In positive characteristic a similar result can be proved using Springer's map [26] between the variety of unipotent elements of G and the variety of nilpotent elements of $\mathfrak{g}$.

Theorem 1.3. *Let G be a reductive algebraic group, P a parabolic subgroup of G, and suppose that* $\operatorname{char} k$ *is a good prime for G. Then* $\operatorname{mod}(P : P_u) = \operatorname{mod}(P : \mathfrak{p}_u)$.

In view of Kirillov's orbit method for the classification of unitary representations of nilpotent Lie groups and in connection with primitive ideals of the enveloping algebra of a complex nilpotent Lie algebra, the coadjoint representation of a Lie group is of particular interest [15, §15], [5].

We consider an algebraic group G and a rational G-module V. Modifying an argument from [21] we show

Theorem 1.4. *Let G be an algebraic group and let V be a rational G-module. Then* $\operatorname{mod}(G : V^*) = \operatorname{mod}(G : V)$. *Moreover, if* $\operatorname{mod}(G : V) = 0$, *then there is a bijection between the sets of G-orbits on V and V^*.*

For a proof see also [17]. As a special case of Theorem 1.4 we have

Corollary 1.5. *Let G be a linear algebraic group and P a parabolic subgroup of G. Then* $\operatorname{mod}(P : \mathfrak{p}_u^*) = \operatorname{mod}(P : \mathfrak{p}_u)$. *Moreover, if* $\operatorname{mod} P = 0$, *then there is a bijection between the sets of P-orbits on $\mathfrak{p}_u$ and $\mathfrak{p}_u^*$.*

Let $\mathcal{T}$ and $\mathcal{T}^*$ be the tangent and cotangent bundle of the flag manifold G/P. As a consequence of Corollary 1.5 we obtain

Corollary 1.6. *Let G be a linear algebraic group and P a parabolic subgroup of G. Then* $\operatorname{mod}(G : \mathcal{T}) = \operatorname{mod}(G : \mathcal{T}^*) = \operatorname{mod} P$. *Moreover, if* $\operatorname{mod} P = 0$, *then there is a bijection between the sets of G-orbits on the two G-bundles.*

2. Notation and Recollection

For an algebraic group G we denote its Lie algebra by $\operatorname{Lie} G$ or by $\mathfrak{g}$, the identity component of G by G^0, and its unipotent radical by G_u. For the Lie algebra of G_u we write $\mathfrak{g}_u$.

Throughout, the identity element of an algebraic group is denoted by e. For $x \in G$ let $\operatorname{Int}(x)$ denote the inner automorphism of G given by conjugation by x. For a subset $S \subset G$ we denote by $C_G(S) := \{g \in G \mid \operatorname{Int}(s)g = g \text{ for all } s \in S\}$ the centralizer of S in G, and likewise by $C_{\mathfrak{g}}(S) := \{Y \in \mathfrak{g} \mid \operatorname{Ad}(s)Y = Y \text{ for all } s \in S\}$ the centralizer of S in $\mathfrak{g}$.

Suppose that G acts morphically on an algebraic variety X. For x in X the G-orbit in X through x is denoted by $G \cdot x$ and for $Z \subset X$ a subset $\cup_{z\in Z} G \cdot z$ by $G \cdot Z$. Recall from [19] that the *modality* of the action of G on X is defined as $\operatorname{mod}(G : X) := \max_Z \min_{z\in Z} \operatorname{codim}_Z G^0 \cdot z$, where Z runs through all irreducible G^0-invariant subvarieties of X. In case X is an irreducible variety let $k(X)^G$ denote the field of G-invariant rational functions on X. As $\min_{x\in X} \operatorname{codim}_X G \cdot x =$