THE RATIONALITY PROBLEM FOR SEMISIMPLE GROUP VARIETIES

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1. Introduction

An irreducible algebraic variety X defined over a field K is called K-rational (resp. stably K-rational) if X (resp. $X \times \mathbb{A}_K^m$ for some $m \in \mathbb{N}$) is birationally isomorphic over K to an affine space. The present paper is devoted to the rationality problem for group varieties. This problem has a long history and is especially interesting for semisimple group varieties.

Let G be a semisimple algebraic group defined over a field K. It is well known that the algebraic variety G is unirational over K, i.e. there exists a dominant rational map $\phi : \mathbb{A}_K^m \to G$ over K or equivalently, the field $K(G)$ of K-rational functions on G is contained in a purely transcendental extension of K (see [DG], Exp. XIV).

Rationality problem. *1) Is the variety G rational over K? In other words, is $K(G)$ a purely transcendental extension of K?*
2) If no, is the variety G then stably rational over K?

If G is K-split (i.e. there exists a maximal torus T splitting over K), then G is K-rational. This follows immediately from the Bruhat decomposition: G is birationally isomorphic over K to the product variety $U \times T \times U$ where U is the unipotent radical of a K-defined Borel subgroup B of G containing T. In particular, over an algebraically closed field K, the variety G is always K-rational.

Other examples of rational group varieties are provided by groups of small ranks. It is known that for any field K all algebraic K-tori of dimension at most two are rational over K (see [V]). This fact combined with the Chevalley-Grothendieck theorem and the K-rationality of the variety of maximal tori of G (see [Ch], [DG], Exp. XIV) implies that G is K-rational if rank $G \leq 2$. In particular, varieties of all K-groups of types A_1, A_2, $B_2 = C_2$, G_2 are rational over K. However in general (i.e. for arbitrary G and K), this problem seems to be difficult. For a long time, the following question was open:

Is the variety G rational over K if G is simply connected?

It turned out that this is not the case. First examples of non-rational varieties of simply connected groups were constructed by the second-named author ([P1]):

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For any m which is not squarefree and any l, there exists a division algebra D of index m over $K = Q(x_1, \dots, x_n)$, or $Q_p(x_1, \dots, x_n)$, $n \geq 2$, such that the variety $G = \mathbf{SL}_l(D)$ is not stably rational over K.

Here Q and Q_p are, respectively, the fields of rational and p-adic numbers and $x_1, \dots, x_n$ are independent variables. In fact, even more was proven: $G(K) = \mathrm{SL}_l(D)$ does not have the weak approximation property.

The following results were inspired by a question posed by P. Deligne. Let f be a non-degenerate quadratic form of rank n over a field K of characteristic $\neq 2$. It is known that the variety $\mathbf{SO}(f)$ is K-rational (see Lemma 5 in §4). In 1978, P. Deligne asked the following question:

Is the spinor variety $\mathbf{Spin}(f)$ rational over K (in particular, over $K = \mathbb{R}$)?

We answered (see [P2], [PC1], [C1]) this question positively in the following cases.

1) If a quadratic form f is K-isotropic, then $\mathbf{Spin}(f)$ is K-rational. In particular, if K is either an algebraic extension of Q_p or a totally imaginary number field or a global function field, then $\mathbf{Spin}(f)$ is K-rational.
2) $\mathbf{Spin}(f)$ is rational over $K = \mathbb{R}$ or Q.
3) $\mathbf{Spin}(f)$ is rational over any K if f is either a sum of squares or a Pfister form.

For an algebraic number field K, which is not totally imaginary, the question of the rationality of $\mathbf{Spin}(f)$ is still open.

Since all groups of rank ≤ 2 are K-rational, the minimal n for which, a priori, there is a possibility of $\mathbf{Spin}(f)$ being non-rational over K, is $n = 6$. The next result was relatively surprising ([P3]):

Let $n = 4m + 2, n \geq 6$, and let $x_1, \dots, x_{n-2}$ be algebraically independent variables over some field L. If $f(y_1, \dots, y_n) = y_1^2 + x_1 y_2^2 + \cdots + x_{n-2} y_{n-1}^2 + x_1 x_2 \cdots x_{n-2} y_n^2$, then the spinor variety $\mathbf{Spin}(f)$ is not stably rational over $K = L(x_1, \dots, x_{n-2})$.

After the discussion above, it becomes clear that it is possible to deal with the question of the rationality of an arbitrary almost simple algebraic K-group only for "good" fields. In the present paper we consider, for instance, the rationality problem for arbitrary almost simple algebraic K-groups over arithmetic fields.

Main Theorem. *Let G be an almost simple algebraic group defined over K, where K is either an algebraic extension of Q_p or a totally imaginary number field or a global function field. If G is not of type $A_n, n \geq 3$, then G is a rational variety over K.*

This statement cannot be extended to groups of type $A_n, n \geq 3$. In fact, now due to A. Merkurjev and M. Rost (see [M1]) we know that the variety $G = \mathbf{SL}_1(D)$, where D is a division algebra of degree $4m$ over any field K, is not rational over K. In particular, this implies that there exists a Q_p-form G of type D_3 such that G is not rational even over Q_p (recall that $D_3 = A_3$), in contrast to the case $D_n,\ n \geq 4$.

If $\mathrm{ind}(D)$ is prime we have the following conjecture which is still open: