On the Cotangent Sheaf of Quot-Schemes

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It is the purpose of this short note to give a *global* description of the cotangent sheaf of Grothendieck's Quot-scheme in terms of a relative Ext-sheaf of the universal subsheaf and the universal quotient sheaf. As an application one gets a description of the cotangent sheaf of the moduli space of stable sheaves in terms of a relative Ext-sheaf involving only a universal family in case such a family exists.

1 Introduction

Let $\mathcal{H}$ be a coherent sheaf on a projective scheme X over an algebraically closed field k. In [1] Grothendieck constructed a projective k-scheme $\mathrm{Quot}(\mathcal{H}, P)$, whose closed points parametrise all possible quotient sheaves F of $\mathcal{H}$ with given Hilbert polynomial P with respect to some fixed ample line bundle $\mathcal{O}_X(1)$. Grothendieck also showed that the Zariski tangent space of $\mathrm{Quot}(\mathcal{H}, P)$ at a closed point $[\rho]$, corresponding to a quotient $\rho : \mathcal{H} \to F$ with kernel $K := \ker(\rho)$, is isomorphic to the space of homomorphisms from K to F:

$$T_{\mathrm{Quot}(\mathcal{H},P)}([\rho]) \cong \mathrm{Hom}(K, F). \tag{1}$$

The question arises how to describe globally the tangent sheaf, or rather the cotangent sheaf, of the Quot-scheme. The answer turns out as follows:

If X is smooth of dimension d and if ω_X denotes the dualising line bundle, we can use Serre duality to rewrite (1) as

$$\mathrm{Ext}^d(F, K \otimes \omega_X) \cong \Omega_{\mathrm{Quot}(\mathcal{H},P)/k}([\rho]). \tag{2}$$

In this form (2) can be globalised: Let $0 \to \mathcal{K} \to \mathcal{O}_{\mathrm{Quot}} \otimes \mathcal{H} \to \mathcal{F} \to 0$ be the universal exact sequence of sheaves on $\mathrm{Quot}(\mathcal{H}, P) \times X$. Let p and q denote the projections from $\mathrm{Quot}(\mathcal{H}, P) \times X$ to $\mathrm{Quot}(\mathcal{H}, P)$ and X, respectively. We claim that there is a natural isomorphism

$$\mathcal{E}xt^d_p(\mathcal{F}, \mathcal{K} \otimes q^*\omega_X) \cong \Omega_{\mathrm{Quot}(\mathcal{H},P)/k}. \tag{3}$$

Note that we do not make any regularity assumptions on $\mathrm{Quot}(\mathcal{H}, P)$. In fact, we will prove this result in the slightly more general framework of a smooth morphism $p : X \to S$ of Noetherian schemes. The main idea for the proof consists in identifying the Quot-scheme as the zero locus of a sheaf homomorphism on an appropriate product of relative Grassmann schemes.

As an application, we get a description of the cotangent sheaf of moduli spaces of stable sheaves: If M is the moduli space of stable sheaves with fixed Hilbert polynomial on the fibres of $X \to S$, and if $\tilde{F}$ is a universal family on $M \times_S X$, then there is an isomorphism $\mathcal{E}xt_p^{d-1}(\tilde{F}, \tilde{F} \otimes \omega) \cong \Omega_{M/S}$.

In order to illustrate the method we begin with the discussion of the easier and also well-known case of the Grassmannian. Afterwards, we first briefly recall Grothendieck's construction of the Quot-scheme and then derive the description of the cotangent sheaf from a more precise formulation of Grothendieck's construction and relative duality.

2 The Grassmannian

Let S be a Noetherian scheme, and let $\alpha : N' \to N$ be a homomorphism of coherent $\mathcal{O}_S$-modules. If N is locally free, then there is a uniquely determined closed subscheme $S_0 \subset S$, the *vanishing locus* of α, such that an arbitrary morphism $g : S' \to S$ factors through S_0 if and only if $g^*(\alpha) = 0$. In fact, if $\alpha' : \mathcal{H}om(N, N') \to \mathcal{O}_S$ is the adjoint of α, then the ideal sheaf $\mathcal{I}_{S_0}$ of S_0 is the image of α'. In particular, if S is a T-scheme, we get an exact sequence $\mathcal{H}om(N|_{S_0}, N'|_{S_0}) \longrightarrow \Omega_{S/T}|_{S_0} \longrightarrow \Omega_{S_0/T} \longrightarrow 0$. This well-known principle can be used to describe the cotangent sheaf of the (relative) Grassmann scheme.

Recall the definition of a Grassmann scheme: If S is a Noetherian scheme, $\mathcal{H}$ a coherent $\mathcal{O}_S$-sheaf and r a positive integer, then there is a projective S-scheme $\pi : G := \mathrm{Grass}(\mathcal{H}, r) \to S$ representing the functor $\underline{\mathrm{Grass}}(\mathcal{H}, r) : (Sch/S) \to (Sets)$, which associates to a morphism $g : T \to S$ the set of all submodules K in $\mathcal{H}_T := g^*\mathcal{H}$ with rank r locally free quotient $\mathcal{H}_T/K$. Let $0 \to A \to \mathcal{H}_G \to B \to 0$ be the universal short exact sequence of $\mathcal{O}_G$-modules.

Let $pr_1, pr_2 : G \times_S G \to G$ denote the projections to the two factors. Any S-morphism $f : T \to G \times_S G$ factors through the diagonal $\Delta \subset G \times_S G$ if and only if the induced quotients $\mathcal{H}_T \to f^* pr_i^* B$ are equivalent, i.e. have the same kernel. Hence the diagonal is the vanishing locus of the composite homomorphism

$$pr_1^* A \longrightarrow \mathcal{H}_{G \times_S G} \longrightarrow pr_2^* B.$$

As B is locally free, we get a natural surjection $\mathcal{H}om(pr_2^* B, pr_1^* A) \longrightarrow \mathcal{I}_\Delta$, which restricted to the diagonal yields a surjection $\varphi_{\mathcal{H}} : \mathcal{H}om(B, A) \longrightarrow \Omega_{G/S}$.

Theorem 2.1 — $\varphi_{\mathcal{H}} : \mathcal{H}om(B, A) \longrightarrow \Omega_{\mathrm{Grass}(\mathcal{H},r)/S}$ *is an isomorphism.*

Proof. Assume first that $\mathcal{H}$ is free of rank n, i.e. $\mathcal{H} \cong \mathcal{O}_S^n$. Then $G \cong S \times_{\mathbb{Z}} \mathrm{Grass}(\mathbb{Z}^n, r)$ is smooth over S of relative dimension $r(n-r)$. Moreover, A is locally free of rank $n - r$. Hence both $\mathcal{H}om(B, A)$ and $\Omega_{G/S}$ are locally free sheaves of the same rank $r(n-r)$. As a surjective homomorphism of locally free sheaves of the same rank $\varphi_{\mathcal{H}}$ must be an isomorphism.

If $\mathcal{H}$ is arbitrary, we may always assume that $\mathcal{H}$ is the quotient of a free sheaf $\mathcal{H}'$, since the question whether $\varphi_{\mathcal{H}}$ is an isomorphism is certainly local in S. The surjection $\rho : \mathcal{H}' \to \mathcal{H}$ induces a closed immersion $G := \mathrm{Grass}(\mathcal{H}, r) \longrightarrow G' := \mathrm{Grass}(\mathcal{H}', r)$. Let $0 \to A' \to \mathcal{H}'_{G'} \to B' \to 0$ be the tautological exact sequence on G'. Then G