Bounds on norms of the adjoint matrix*

Ludwig Elsner
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D-33615 Bielefeld
Germany

Daniel Hershkowitz
Department of Mathematics
Technion - Israel Institute of Technology
Haifa 32000
Israel

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ABSTRACT

An upper bound on operator norms of the adjoint matrix is presented, and special cases of the l_1, l_2 and l_∞ norms are investigated. The results are then used to obtain lower bounds on the smallest absolute value of an eigenvalue of a nonsingular matrix.

1 Introduction

For a matrix A in the vector space C^{nn} of all $n \times n$ complex matrices we denote by $\lambda_1(A), \ldots, \lambda_n(A)$ the eigenvalues of A ordered in a non-increasing order of their moduli. Clearly, the value of $|\lambda_n(A)|$ can serve as a kind of measure for the nonsingularity of A. In this paper we find lower bounds on $|\lambda_n(A)|$.

For this purpose, we consider an absolute vector norm $||.||$ on C^n. For a matrix A in C^{nn} we denote by $||A||$ the corresponding operator norm. Also, we denote by $adj(A)$ the (classical) adjoint matrix of A, that is, the transposed matrix of cofactors. Note that the term *adjugate* is sometimes used instead of *adjoint* to avoid confusion with the Hermitian adjoint A^*.

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We define a constant $\kappa(||.||)$ associated with the operator norm, and we present an upper bound on $||adj(A)||$ in terms of the constant $\kappa(||.||)$ and the vector norms of the rows of A. We apply this result to the norms l_1, l_2 and l_∞, and we use the resulting inequalities to derive our lower bounds on $|\lambda_n(A)|$.

2 Upper bounds on norms of the adjoint matrix

Let $||.||$ be an absolute vector norm on C^n, and for a matrix A in C^{nn} let $||A||$ be the corresponding operator norm. We define

$$\kappa(||.||) = max\{||adj(B)|| : B \in C^{nn},\ ||row_i(B)|| = 1,\ i = 1, \ldots, n\},$$

where $row_i(B)$ denotes the (vector in C^n which is the) transpose of the ith row of B.

The following theorem is the main tool from which we derive our results.

Theorem 2.1 *Let $A \in C^{nn}$. We have*

$$||adj(A)|| \leq \kappa(||.||) \max_k \prod_{\substack{j=1 \\ j \neq k}}^{n} ||row_j(A)||. \tag{2.2}$$

Proof. Assume first that A has no zero rows. Note that for every nonsingular matrix R we have

$$adj(R^{-1}A)adj(R) = adj(A).$$

In particular, if we choose the matrix R to be the diagonal matrix $diag(r_i)$ where $r_i = ||row_i(A)||$, $i = 1, \ldots, n$, then

$$||adj(R)|| = ||r_1 r_2 \cdots r_n diag(r_i^{-1})|| = \max_k \prod_{\substack{j=1 \\ j \neq k}}^{n} ||row_j(A)||,$$

and since $R^{-1}A$ has norm 1 rows, it now follows that

$$||adj(A)|| \leq ||adj(R^{-1}A)||\, ||adj(R)|| \leq \kappa(||.||) \max_k \prod_{\substack{j=1 \\ j \neq k}}^{n} ||row_j(A)||.$$

If A has zero rows then we apply the above procedure to the matrix $A + \epsilon I$ for ϵ sufficently small, and then use continuity arguments to prove our assertion. $\square$

In the rest of this section we evaluate $\kappa(||.||_p)$ for the l_p norm $||.||_p$, $p = 1, 2, \infty$.