Non-bicritical Critical Snarks

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Abstract

Snarks are cubic graphs with chromatic index $\chi' = 4$. A snark G is called critical if $\chi'(G - \{v, w\}) = 3$ for any two adjacent vertices v and w, and it is called bicritical if $\chi'(G - \{v, w\}) = 3$ for any two vertices v and w. We construct infinite families of critical snarks which are not bicritical. This solves a problem stated by Nedela and Škoviera in [7].

1 Introduction

We are using standard graph theoretical terminology and notation in this paper. We define a *snark* to be a cubic graph with chromatic index $\chi' = 4$.
The study of reduction of snarks is as old as the study of snarks itself. The first reductions which are considered were edge-cut reduction and decomposition of snarks, cf. [2, 5, 6, 7, 8, 11, 12]. In [7] Nedela and Škoviera gave these two concepts a precise meaning, and we refer the reader to this paper for a more extensive introduction. In that paper the reduction of snarks is defined as follows.
Let $G = (V(G), E(G))$ be a snark and let $F \subset E(G)$ be an k-edge cut ($k \geq 0$) whose removal divides G into two components H_1 and H_2. If the chromatic index of one of the components is 4, say $\chi'(H_1) = 4$, then H_1 can be extended to a snark $H = (V(H), E(H))$ with $|V(H)| \leq |V(G)|$, and H is called a *k-reduction* of G. If $|V(H)| < |V(G)|$ then H is called a *proper* k-reduction of G.
A snark is called *k-irreducible* if it has no proper m-reduction for each $m < k$, and it is called *irreducible* if it k-irreducible for each $k \geq 1$.
A snark G is called *critical* if $\chi'(G - \{v, w\}) = 3$ for any two adjacent vertices $v, w \in V(G)$, and it is called *cocritical* if $\chi'(G - \{v, w\}) = 3$ for any two nonadjacent vertices v and w of G. Finally, a snark is called *bicritical* if it is critical and cocritical.
Nedela and Škoviera proved the following characterizations.

Theorem 1.1 ([7]) *Let G be a snark. Then the following statements hold true.*

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1. *If $5 \leq k \leq 6$, then G is k-irreducible if and only if it is critical.*
2. *If $k \geq 7$, then G is k-irreducible if and only if it is bicritical.*

Finally it turns out that

Theorem 1.2 ([7]) *A snark is irreducible if and only if it is bicritical.*

In [10] the author showed that there are bicritical of order n for each even $n \geq 92$. In that paper there is also given another characterization of bicritical snarks that yields a method for constructing all snarks starting from the bicritical ones.
In [1] it is shown that there are cocritical snarks which are not critical, and that there are snarks which are neither critical nor cocritical. Further there is no snark of order smaller than 30 that is critical and not cocritical, i.e. each critical snark on less than 30 vertices is bicritical and henceforth it is irreducible.
Nedela and Škoviera [7] state the following Problem.

Problem 1.3 ([7]) *Does there exist a snark that is critical but not bicritical? Equivalently, does there exist a 6-irreducible snark that is not irreducible?*

The main result of this paper is the following theorem, which answers the question above in the affirmative.

Theorem 1.4 *There are infinitely many critical snarks which are not bicritical.*

As a corollary we state

Corollary 1.5 *There are critical snarks which are not bicritical on n vertices for $n = 32$ and for each $n \in \{8k+2 | k \geq 5\}$.*

To prove these two statements we construct infinite families of critical snarks which are not bicritical.

2 Construction of non-bicritical critical Snarks

The 7-block B_i is the graph obtained from the cycle C_6 on the vertices $v_{7i}, v_{7i+1}, \ldots, v_{7i+5}$ by adding a vertex v_{7i+6} and edges $v_{7i+6}v_{7i+1}$ and $v_{7i+6}v_{7i+4}$.
The Goldberg-graph F is obtained from three 7-blocks B_0, B_1 and B_2 by adding a vertex v_{21} and edges $v_{21}v_6, v_{21}v_{13}, v_{21}v_{20}$ and $v_0v_{12}, v_3v_9, v_7v_{19}, v_{10}v_{16}$. The graph is drawn in Figure 1.
Goldberg [5] showed that F is a 3-critical graph, where a graph is called *k-critical* ($k \geq 1$) if $\Delta(G) = k$, $\chi'(G) = k + 1$ and $\chi'(G - e) < \chi'(G)$ for each edge e of G. It is well known that each 3-critical graph contains a vertex of degree 2.