Cofibration and Fibration Structures in Enriched Categories

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0 Introduction

This paper is motivated by the attempt to lay out the grounds for relative homological algebra for module spectra. Elmendorf, Kriz, Mandell, and May constructed an associative, commutative and unital smash product for spectra with some additional structure, called S-module spectra [6]. This good smash product allows the translation of concepts of classical homological algebra to the category ${}_S\mathcal{M}od$ of S-module spectra. Since ${}_S\mathcal{M}od$ is not abelian, the standard approach to derived functors does not apply. The most common frame work for doing homological algebra in the non-abelian case is Quillen's concept of a closed model category sytructure. Elmendorf, Kriz, Mandell and May construct a topological closed model structure on ${}_S\mathcal{M}od$ whose weak equivalences are maps inducing isomorphisms on homotopy groups and whose cofibrations are retracts of relative cell-objects. Relative cell-objects are constructed analogous to relative CW-complexes.

Relative homological algebra deals with weak equivalences which are genuine homotopy equivalences after forgetting part of the structure. Genuine homotopy equivalences often cannot be tested by a "set of small objects", and in the absence of small objects arguments one has trouble to define closed model structures.

Fortunately, one only needs parts of a closed model structure to define left or right derived functors: cofibration and fibration structures suffice [1], [13], and genuine homotopy equivalences fit nicely into such structures.

One of the most useful technical results for cofibrations of topological spaces is the "product theorem": if $i : A \to X$ and $j : B \to Y$ are cofibrations and one of them is closed then

$$(i \times Y, X \times j) : A \times Y \cup X \times B \longrightarrow X \times Y$$

is a cofibration. The condition that i or j is closed indicates that the proof of this result is not formal. In the absence of a product theorem alternative reasoning is necessary where-ever this result has been used in the corresponding topological case (e.g. see [6, VII. 6.8]).

Taking one of the properties of a closed cofibration as an axiom we introduce the notion of a strong cofibration. It turns out that the cofibrations in the closed model category structures of the categories of spectra defined by Elmendorf, Kriz, Mandell, and May are strong ones, and strong cofibrations are less rigid than relative cell objects: e.g. various forgetful functors map relative cell objects to strong cofibrations even in cases where relative cell objects are not preserved (see §6). Moreover, we derive a pairing theorem for strong cofibrations (Thm. 2.11), which specializes to the product theorem in the topological case.

Although this result holds more generally we prove it in the setting of enriched categories which are complete and cocomplete, tensored and cotensored.

The pairing theorem then is used to verify the axioms for fibrations and cofibration structures on our categories with genuine homotopy equivalences as weak equivalences.

We conclude with a list of categories which can be used for enriching, an analysis of cofibrations in categories of spectras and a few other examples to which the results can be applied.

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1 Enriched categories

To specify notations we recall the basic concepts of enriched category theory from [10].

Let $\mathcal{V} = (\mathcal{V}_0, \Box, \Phi)$ be a symmetric monoidal category with unit object Φ. We assume $\mathcal{V}$ to be *closed*, i.e. there is a bifunctor

$$[-,-] : \mathcal{V}_0^{op} \times \mathcal{V}_0 \to \mathcal{V}_0$$

satisfying the Hom-tensor adjunction, i.e. there is a natural isomorphism

1.1

$$\mathcal{V}_0(X \Box Y, Z) \cong \mathcal{V}_0(X, [Y, Z])$$

From these data we can deduce natural isomorphisms

1.2 (1) $\mathcal{V}_0(Y, Z) \cong \mathcal{V}_0(\Phi \Box Y, Z]) \cong \mathcal{V}_0(\Phi, [Y, Z])$

(2) $Z \cong [\Phi, Z]$

(3) $[X \Box Y, Z] \cong [X, [Y, Z]]$

As a sample we prove (3). For each A in $\mathcal{V}_0$ we have a sequence of natural bijections

$$\mathcal{V}_0(A, [X \Box Y, Z]) \cong \mathcal{V}_0(A \Box X \Box Y, Z]) \cong \mathcal{V}_0(A \Box X, [Y, Z]) \cong \mathcal{V}_0(A, [X, [Y, Z]])$$

which implies the result.