Reduction of Symmetric Configurations n_3

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Abstract

Symmetric configurations n_3 are the bicubic graphs of girth ≥ 6. They are all reducible to the configuration 7_3, the Fano-Heawood-graph F i.e. the projective plane of order 2 by means of a Martinetti-like procedure.

1 Introduction

A *configuration* (v_r, b_k) is an incidence structure with the following properties:

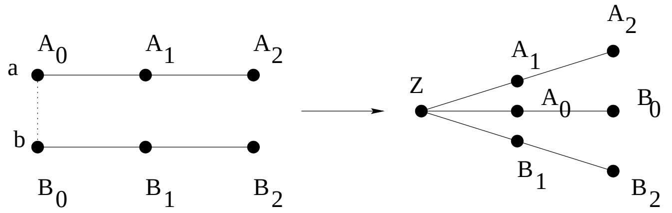
(a) There are v points and b lines.

(b) There are k points on each line and r lines through each point.

(c) Two different lines intersect at most once and two different points are connected by at most one line.

If $v = b$ (and hence $r = k$) then the configuration is called *symmetric* and denoted by v_k.

Configurations were defined by Reye [2] in 1876. In 1881 Kantor enumerated all symmetric configurations n_3 for $n \leq 10$. In 1887 Martinetti [1] gave a construction method for the symmetric configurations n_3.
Given a configuration n_3. If possible, take away two parallel (non intersecting) lines a and b and two points A_0 and B_0 from a and b resp. which are not connected. Remove the lines a and b and add a new point Z and three new lines $\{Z, A_0, A_2\}, \{Z, B_1, B_2\}, \{Z, A_0, B_0\}$ through Z. A configuration $(n+1)_3$ is obtained.



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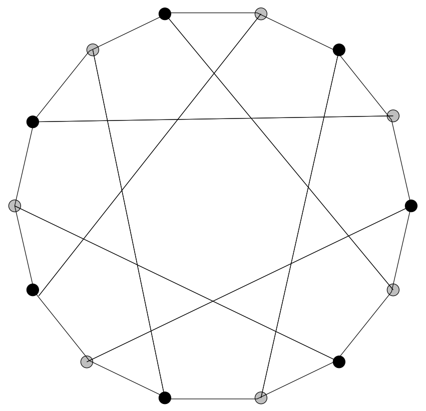
This is not always possible. Martinetti called the configurations n_3 constructed in this way *reducible*. His method does not cover an infinite number of exeptions, called *irreducible*.

Some famous configurations n_3 are the Fano-plane 7_3, the Pappus Configuration 9_3 and the Desargues Configuration 10_3, which are important in projective and affine geometry. There is an interpretation of configurations as graphs. We could think of the lines as white vertices and the points as black ones. Two vertices are connected by an edge, if the corresponding line and point are incident. Since two lines intersect at most once and two points are connected by at most one line, the graph contains no cycle of length 4. The girth is greater or equal to 6. We will identify configurations and their graphs.
Symmetric configurations n_k are k-regular bipartite graphs. The configurations n_3 which we are interested in are 3-regular bipartite graphs, so called bicubic graphs, with girth ≥ 6. We give an example of this interpretation for the Fano-plane 7_3 which is the Heawood graph F.



Fano-Heawood Graph F

Martinetti's result with the infinity of exceptions motivated the development of a generalized reduction process by means of which every n_3 can be reduced to F. In this paper we shall prove the following theorem:

Theorem: There exists a finite number of generalized Martinetti-like reduction processes, such that every bicubic connected graph G with girth ≥ 6 different from the Fano-Heawood graph F can be reduced to a smaller one of the same kind, having two vertices less, and by iteration to F.

2 Symmetric Configurations n_3

We give a reduction procedure for the class of connected finite bicubic graphs with girth ≥ 6. Using this procedure every graph n_3 can be reduced to a graph $(n-1)_3$, as long as it is not in the base, the set of minimal elements. It is well