The preprojective algebra of a quiver

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ABSTRACT. The preprojective algebra $\mathcal{P}_k(Q)$ of a quiver Q plays an important role in mathematics. We are going to present some descriptions of these algebras and their module categories which seem to be well-accepted by the experts, but for which we were unable to find complete proofs in the literature. In particular, we determine the fibre of the forgetful functor from the category of $\mathcal{P}_k(Q)$-modules to the category of kQ-modules in terms of the orbit algebra of a kQ-module with respect to the Auslander-Reiten translation. The aim of the present paper is to outline in which way the representation theory of kQ can be used in order to deal with varieties studied by Lusztig.

1. Introduction

Let k be a field. The algebras which we consider will be k-algebras, modules are usually left modules. If A is a k-algebra, we denote by $\operatorname{Mod} A$ the category of all A-modules, by $\operatorname{mod} A$ the full subcategory of all finite dimensional A-modules.

Quivers. Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver; here, Q_0, Q_1 are finite sets, and $s, t\colon Q_1 \to Q_0$ are maps; the elements of Q_0 are called *vertices*, those of Q_1 are called *arrows*; given an arrow $\alpha \in Q_1$, one writes $\alpha\colon s\alpha \to t\alpha$ and calls $s\alpha$ its starting vertex and $t\alpha$ its terminal vertex. A path of length $l \geq 1$ is a sequence $(\alpha_1, \dots, \alpha_l)$ with $s\alpha_i = t\alpha_{i+1}$, for $1 \leq i < l$; the vertex $s\alpha_l$ is called its starting vertex and $t\alpha_1$ is called its terminal vertex; in addition, one also considers paths of length 0, they correspond bijectively to the vertices of Q. We denote by kQ the path algebra of the quiver Q with coefficient field k; here, the product of two paths is given by concatenation, whenever this is possible, and by zero otherwise. Note that the path algebra kQ has global dimension at most 1. Of course, kQ is finite dimensional if and only if there are no cyclic paths in Q (a cylic path is a path of length at least 1 with same starting and terminal vertex). Let kQ^+ be the subspace of kQ with basis the set of all paths of length at least 1; it is the ideal of kQ generated by the arrows.

The preprojective algebra of the quiver Q. Let $\overline{Q}$ be obtained from Q by adding for every arrow $\alpha\colon x \to y$ a formal inverse $\alpha^*\colon y \to x$, the set of new arrows will be denoted by Q_1^*. If Q has loops, then, by construction, the number of loops of $\overline{Q}$ is twice the number of loops of Q. (We may characterize $\overline{Q}$ as follows: it is a finite quiver with a fixpoint free involution ι on the set of arrows, such that for every arrow β of $\overline{Q}$ the starting point of $\iota(\beta)$ is the endpoint of β or, equivalently, the compositions $\beta\iota(\beta)$ and $\iota(\beta)\beta$ are defined; the quiver Q contains precisely one arrow in each ι-orbit of $\overline{Q}$.) We define on Q_1 an integer valued function $\varepsilon_Q\colon Q_1 \to \mathbb{Z}$

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by

$$\varepsilon_Q(\beta) = \begin{cases} 1 & \text{provided } \beta \in Q_1, \\ -1 & \text{provided } \beta \in Q_1^*. \end{cases}$$

We consider the following element

$$\rho = \sum_{\alpha \in \overline{Q}_1} \varepsilon_Q(\alpha)\alpha^*\alpha$$

in $k\overline{Q}$ and the ideal (ρ) generated by ρ. The algebra $\mathcal{P}_k(Q) = k\overline{Q}/(\rho)$ will be called the *preprojective algebra of the quiver Q*.

Tensor algebras of bimodules. If Λ is a ring and Ω a Λ-bimodule, let $\Lambda\langle\Omega\rangle$ denote the corresponding tensor algebra; it is the direct sum

$$\Lambda\langle\Omega\rangle = \bigoplus_{t \geq 0} \Omega^{\otimes t},$$

where $\Omega^{\otimes t}$ is the t-fold tensor power of Ω, with $\Omega^{\otimes 0} = \Lambda$, and $\Omega^{\otimes(t+1)} = \Omega^{\otimes t} \otimes \Omega$; the product of $a \in \Omega^{\otimes s}$ and $b \in \Omega^{\otimes t}$ in the tensor algebra is just $a \otimes b \in \Omega^{\otimes(s+t)}$, provided $s, t \geq 1$, and the scalar product ab otherwise. Note that Λ is a subring of $\Lambda\langle\Omega\rangle$, thus there is a forgetful functor from the category of all $\Lambda\langle\Omega\rangle$-modules to the category of Λ-modules. The ideal of $\Lambda\langle\Omega\rangle$ generated by Ω is called the *augmentation ideal.*

We denote by $D = \operatorname{Hom}_k(-, k)$ the usual k-duality: for example, starting with the right module kQ_{kQ}, we obtain the dual module $D(kQ_{kQ})$, and we may consider $\operatorname{Ext}^1_{kQ}(D(kQ_{kQ}), {}_{kQ}kQ)$; this is a kQ-bimodule: we use as left module structure the canonical action of kQ on the right of $D(kQ_{kQ})$, and as right module structure the the canonical action of kQ on the right of ${}_{kQ}kQ$.

Theorem A. *Let Q be a quiver without cyclic paths. Let*

$$\Theta = \operatorname{Ext}^1_{kQ}(D(kQ_{kQ}), {}_{kQ}kQ).$$

The algebras $\mathcal{P}_k(Q)$ and $kQ\langle\Theta\rangle$ are isomorphic.

There exists an isomorphism whose restriction to kQ is the identity and which maps the ideal of $\mathcal{P}_k(Q)$ generated by the arrows of Q^* onto the augmentation ideal of $kQ\langle\Theta\rangle$.

Both algebras $\mathcal{P}_k(Q)$ and $kQ\langle\Theta\rangle$ have been studied by many mathematicians. The aim of Gelfand and Ponomarev [GP] was to construct an algebra A with the following property $(*)$: it contains kQ as a subalgebra and when considered as a left kQ-module, A decomposes as a direct sum of the indecomposable 'preprojective' kQ-modules, one from each isomorphism class (the definition of preprojective modules will be recalled in section 5 of the paper). Our joint paper [DR] with Dlab had the same aim in mind, but dealt with the more general case of a given species instead of a quiver. Since the algebra $\mathcal{P}_k(Q)$ has the property $(*)$, it became customary to call it the preprojective algebra of the quiver Q. But let us stress that for a fixed quiver Q, there may be several isomorphism classes of algebras A with property $(*)$, see the remark at the end of the paper. Algebras of the form $\mathcal{P}_k(Q)$ appear quite