MODERATE AND LARGE DEVIATIONS FOR U-PROCESSES

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ABSTRACT. Sufficient conditions for the moderate and large deviation principle for U-processes are given. For the large deviation result the conditions are in terms of "blockwise" empirical process conditions. On the moderate scaling the case of U-processes indexed by a uniformly bounded VC subgraph class of functions is considered. The proofs are based on an isoperimetric inequality for empirical processes due to Talagrand, a truncation method based on an isoperimetric inequality by Ledoux, the existence of almost regular partitions of complete hypergraphs due to Baranyai and a Bernstein-type inequality for U-processes due to Arcones and Giné.

1. INTRODUCTION

Let $(S, \mathcal{S}, \mu)$ be a probability space and let $X_i : S^{\mathbb{N}} \to S$ be the coordinate functions ($\{X_i\}_{i\in\mathbb{N}}$ is thus an i.i.d. sequence with $\mathcal{L}(X_i) = \mu$). The U-empirical measure of order m is defined by

$$L_n^m := \frac{1}{n_{(m)}} \sum_{(i_1,\dots,i_m)\in I_{m,n}} \delta_{(X_{i_1},\dots,X_{i_m})},$$

where $n_{(m)} := \prod_{k=0}^{m-1}(n-k)$, $I_{m,n} \subset \{1,\dots,n\}^m$ contains all m-tuples with pairwise different components and δ_x denotes the probability measure degenerate at $x \in S$. Let $\mathcal{H}$ be a collection of measurable functions $h : S^m \to \mathbb{R}$. The U-process of order m indexed by $\mathcal{H}$ is defined, for every $n \in \mathbb{N}$, as

$$U_n^m(h,\mu) := \int_{S^m} h\, dL_n^m = \frac{1}{n_{(m)}} \sum_{(i_1,\dots,i_m)\in I_{m,n}} h(X_{i_1},\dots,X_{i_m}), \quad h \in \mathcal{H}.$$

For a fixed h, $U_n^m(h,\mu)$ is called a U-statistic of order m with kernel h based on the probability mearsure μ. U-processes appear in statistics for example as unbiased estimators of the functional $\{\mu^{\otimes m} h : h \in \mathcal{H}\}$. For instance, Liu's simplicial depth process (Liu 1990) is a U-process. Noland and Pollard (1987), (1988) studied the law of large numbers and the central limit theorem for U-processes of order $m = 2$. Arcones and Giné (1991) developed the theory for an arbitrary m. An overview over the theory of empirical processes and U-processes is Giné (1996). Wu (1994) proved necessary and sufficient conditions for the large deviation and moderate deviation

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estimations and the LIL of the empirical process $L_n(h) = 1/n \sum_{i=1}^n h(X_i)$ with h varying in a class of bounded functions. The principles are proved for laws in the Banach space of bounded functionals on the class $\mathcal{H}$.

In this paper we study the large and moderate deviation principles (LDP, MDP) for U-processes under certain conditions on $\mathcal{H}$.

Let us recall the definition of the LDP. A sequence of probability measures $\{\mu_n\}_{n \in \mathbb{N}}$ on a topological space $\mathcal{X}$ equipped with σ-field $\mathcal{B}$ is said to satisfy the LDP with speed $a_n \downarrow 0$ and good rate function $I(\cdot)$ if the level sets $\{x : I(x) \leq \alpha\}$ are compact for all $\alpha < \infty$ and for all $\Gamma \in \mathcal{B}$ the lower bound

$$\liminf_{n\to\infty} a_n \log \mu_n(\Gamma) \geq - \inf_{x \in \mathrm{int}(\Gamma)} I(x),$$

and the upper bound

$$\limsup_{n\to\infty} a_n \log \mu_n(\Gamma) \leq - \inf_{x \in \mathrm{cl}(\Gamma)} I(x)$$

hold, where $\mathrm{int}(\Gamma)$ and $\mathrm{cl}(\Gamma)$ denote the interior and closure of Γ, respectively. We shall consider situations where the speed in the LDP is given by $a_n = 1/n$. We say that a sequence of random variables satisfies the LDP when the sequence of measures induced by these variables satisfies the LDP. Let $\{b_n\}_{n\in\mathbb{N}} \subset (0, \infty)$ be a sequence which satisfies

$$\lim_{n\to\infty} \frac{b_n}{n} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{n}{b_n^2} = 0. \tag{1.1}$$

If $a\,x \in \mathcal{X}$ for all $a > 0$ and $x \in \mathcal{X}$ then a sequence of random variables Z_n shall satisfy the MDP with speed $\frac{n}{b_n^2}$ and with good rate function $I(\cdot)$, if the sequence $\frac{n}{b_n} Z_n$ satisfies the LDP in $\mathcal{X}$ with the good rate function $I(\cdot)$ and with speed $\frac{n}{b_n^2}$.

As the best of our knowledge, the present situation of the LDP and MDP problem for U-statistics and U-empirical measures is as follows. Let $\mathcal{M}_1(S^m)$ denote the space of Borel probability measures on S^m. The LDP for the sequence of laws of L_n^m has been studied in Eichelsbacher and Schmock (1996). Let $B(S^m, E)$ denote the space of bounded functions on S^m, which take values in a separable real Banach space $(E, ||\cdot||_E)$ with Borel σ-algebra $\mathcal{E}$ and are $\mathcal{S}^{\otimes m}$-$\mathcal{E}$-measurable. The LDP for the sequence of laws of L_n^m holds on $\mathcal{M}_1(S^m)$ equipped with the $\tau_1(E)$-topology which makes the maps $\mathcal{M}_1(S^m) \ni \nu \mapsto \int_{S^m} \varphi \, d\nu$ continuous for all $\varphi \in B(S^m, E)$, where $\int$ denotes the Bochner integral. The speed is $1/n$ and the rate function is given by

$$J_m(\nu) = \begin{cases} \frac{1}{m} H_m(\nu \,|\, \mu^{\otimes m}), & \text{if } \nu = \nu_1^{\otimes m}, \\ \infty & \text{otherwise}, \end{cases} \tag{1.2}$$

where ν_1 denotes the first marginal of ν, and

$$H_m(\nu \,|\, \tilde{\nu}) = \begin{cases} \int_{S^m} f \log f \, d\tilde{\nu}, & \text{if } \nu \ll \tilde{\nu} \text{ and } f = \frac{d\nu}{d\tilde{\nu}}, \\ \infty & \text{otherwise} \end{cases}$$