Analysis and geometry on configuration spaces

S. Albeverio
Fakultät für Mathematik, Ruhr–Universität,
D 44780 Bochum (Germany)

Yu. G. Kondratiev
Forschungszentrum BiBoS, Bielefeld Universität,
D 33615 Bielefeld (Germany);
Institute of Mathematics, NASU,
252601 Kiev (Ukraine)

M. Röckner
Fakultät für Mathematik, Bielefeld Universität,
D 33615 Bielefeld (Germany)

Abstract

In this paper foundations are presented to a new systematic approach to analysis and geometry for an important class of infinite dimensional manifolds, namely, configuration spaces. More precisely, a differential geometry is introduced on the configuration space Γ_X over a Riemannian manifold X. This geometry is "non-flat" even if $X = \mathbb{R}^d$. It is obtained as a natural lifting of the Riemannian structure on X. In particular, a corresponding gradient ∇^Γ, divergence div^Γ, and Laplace-Beltrami operator $H^\Gamma = -\mathrm{div}^\Gamma \nabla^\Gamma$ are constructed. The associated volume elements, i.e., all measures μ on Γ_X w.r.t. which ∇^Γ and div^Γ become dual operators on $L^2(\Gamma_X;\mu)$, are identified as exactly the mixed Poisson measures with mean measure equal to a multiple of the volume element dx on X. In particular, all these measures obey an integration by parts formula w.r.t. vector fields on Γ_X. The corresponding Dirichlet forms $\mathcal{E}_\mu^\Gamma$ on $L^2(\Gamma_X;\mu)$ are, therefore, defined. Each is shown to be associated with a diffusion process which is thus the Brownian motion on Γ_X and which is subsequently identified as the usual independent infinite particle process on X. The associated heat semigroup $(T_\mu^\Gamma(t))_{t>0}$ is calculated explicitly. It is also proved that the diffusion process, when started with μ, is time-ergodic (or equivalently $\mathcal{E}_\mu^\Gamma$ is irreducible or equivalently $(T_\mu^\Gamma(t))_{t>0}$ is ergodic) if and only if μ is Poisson measure $\pi_{z\,dx}$ with intensity $z\,dx$ for some $z \geq 0$. Furthermore, it is shown that the Laplace-Beltrami operator $H^\Gamma = -\mathrm{div}^\Gamma \nabla^\Gamma$ on $L^2(\Gamma_X;\pi_{z\,dx})$ is unitary equivalent to the second quantization of the Laplacian $-\triangle^X$ on X on the corresponding Fock space $\bigoplus_{n\geq 0} L^2(X;z\,dx)^{\hat{\otimes} n}$. As another direct consequence of our results we obtain a representation of the Lie-algebra of compactly supported vector fields on X on Poisson space. Finally, generalizations to the case where dx is replaced by an absolutely continuous measure and also to interacting particle systems on X are described, in particular, the case where the mixed Poisson measures μ are replaced by Gibbs measures of Ruelle-type on Γ_X.