# Strong symmetry defined by twisting modules, applied to quasi–hereditary algebras with triangular decomposition and vanishing radical cube

**Steffen König and Changchang Xi**

**Abstract**

For a finite dimensional algebra with triangular decomposition, a new kind of modules is defined, the twisting modules. Using the structure of these modules, for some algebras with vanishing radical cube the characteristic tilting module is described and its endomorphism ring is computed. This covers both Temperley–Lieb algebras and $q$–Schur algebras of finite representation type. Our basic tool is a new symmetry condition, stronger than the symmetry provided by the existence of a triangular decomposition in general.

## 1 Introduction

Finite dimensional quasi–hereditary algebras occur rather frequently in applications: (classical or quantized) Schur algebras whose modules are the polynomial representations of (classical or quantum) $GL_n$ have this structure as well as the algebras associated with blocks of the BGG–category $\mathcal{O}$ of a semisimple complex Lie algebra and as Temperley–Lieb algebras (see [12]) which are of interest both in statistical mechanics and for constructing the Jones knot polynomial. Triangular decompositions for finite dimensional algebras are the analogue of PBW theorems for (quantized) enveloping algebras and also of crossed products (e.g. the Drinfeld double) for Hopf algebras. Such decompositions exist in many of the situations just mentioned.

Various questions of interest in applications can be asked for the class of algebras with triangular decompositions, or more generally, for quasi–hereditary algebras. A basic problem is to compute quiver and relations of such algebras. This is needed for example in order to apply the sophisticated methods of representation theory of finite dimensional algebras. For many of the algebras mentioned above, the quiver (which has as vertices the isomorphism classes of simple modules and arrows corresponding to extensions between simples) can be computed. For example in the case of category $\mathcal{O}$, knowing the quiver is just an equivalent form of the (proven) Kazhdan–Lusztig conjecture. However, no general methods seem to be known to compute the relations.

Moreover, one would like to be able to compute Ringel's characteristic tilting modules, their endomorphism rings, and the quadratic duals of algebras with triangular decompositions. However, at present these questions seem to be too hard for this quite large class of algebras in general. Thus there are two possible strategies. One strategy is to look for additional properties, in particular symmetry conditions, which are satisfied in (most of) the applications in order to make the class of algebras smaller. The other one is to study subclasses which are easier to access, but contain at least reasonably many examples from applications. As examples of related investigations we mention Dyer's study of blocks of category $\mathcal{O}$ and related algebras [7, 8], and the equivalences Khoroshkin has found between certain categories of $D$–modules and certain representations of quivers [13].

In this note we study such a subclass. We require the algebras to be quadratic and — a strong restriction — to have vanishing radical cube. This covers all Temperley–Lieb algebras, and a few of the other applications (for example, (quantized) Schur algebras of finite representation type). We give a general construction by quiver and relations of the algebras in this class, and find a condition, satisfied for Temperley–Lieb algebras, under which we prove that the characteristic tilting module is induced from an injective module over the exact Borel subalgebra. (Note that this is not true for the dual extensions defined in [25] and studied in [4]). In this way we compute the characteristic tilting modules and determine their endomorphism rings. We also compute quadratic duals. Verifying the condition for Temperley–Lieb algebras, we also classify the blocks of these algebras.

The condition which we use and which is satisfied in all these examples is an isomorphism between certain induced and certain coinduced modules. This seems to be a 'strong symmetry condition' of the kind one has to look for when following the first of the above strategies.

Such symmetry conditions are also of interest with respect to the following **uniqueness problem** (which at present is completely open): A finite dimensional algebra $A$ having a triangular decomposition $A \simeq C \otimes_S B$ is not uniquely determined by the data $C, B, S$. In view of the analogy with crossed products for Hopf algebras one would like to find a special such algebra $A$, analogous to the Drinfeld double, with better symmetry properties. Our strong symmetry condition might be such a symmetry property (see propositions 3.3, 3.4, and section 5, where we use strong symmetry to classify the blocks of Temperley–Lieb algebras).

Our main tool is what we call twisting modules. They are defined in section three (for algebras with triangular decomposition in general). In section four we look at the related concept of twisting relations. In section five we formulate our 'strong symmetry condition'. In section six we give a general construction of algebras with triangular decomposition. In section seven we classify the blocks of Temperley–Lieb algebras. In section eight we produce characteristic tilting modules by induction from an exact Borel subalgebra, and we compute their endomorphism rings. Quadratic duals are computed in section nine. Proofs of all results are given in section ten.