Factorizations of primitive permutation groups.

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1 Introduction.

In this paper we attack the following problem in the theory of permutation groups

> Classify all finite primitive permutation groups (G, Ω) which admit a factorization $G = G_\omega G_\omega^\alpha$ in a point stabilizer G_ω and the image G_ω^α of G_ω under an automorphism α of G.

We will call such a factorization of a group a *primitive factorization*.

We are interested in primitive factorizations, since we want to determine the primitive flag–transitive $m \times m$ grids (Γ, X). Here primitive means that the stabilizer G in the group X of one of the two parallel classes of lines of Γ, say Π_1, acts primitively on Π_1.

A primitive flag–transitive grid gives rise to a primitive factorization and conversely: If (Γ, X) is a flag–transitive $m \times m$ grid, then X contains a subgroup G of index 2 that admits a factorization $G = G_l G_l^\alpha$, where G_l is the stabilizer of a line l and α an element of $X \setminus G$. The factorization $G = G_l G_l^\alpha$ is primitive if and only if (Γ, X) is a primitive grid.

On the other hand suppose that a group G admits a factorization $G = G_\omega G_\omega^\alpha$ for an $\alpha \in \mathrm{Aut}(G)$ with $\alpha^2 \in G$. Then the coset geometry Γ with set of points being the cosets of $(G_\omega \cap G_\omega^\alpha)\langle\alpha\rangle$ in $X = G\langle\alpha\rangle$ and set of lines being the cosets of G_ω in X supplies us with a $m \times m$ grid, where $m = |G_\omega : G_\omega \cap G_\omega^\alpha|$, with the group X acting flag–transitively.

For geometric purposes it is natural and usual to assume primitivity. Often a problem on flag-transitive geometries can be reduced to a problem where a group is acting primitively in some way. This happens for instance in the classification of linear spaces, see [BueDeDo], or in the classification of distance transitive-graphs, see [Sm]. In fact the classification of flag-transitive grids can be reduced to this one of primitive flag-transitive grids: Any imprimitive flag-transitive grid (Γ, X) gives rise to a primitive flag-transitive grid (Γ', X), see Lemma 1.5.

Recall that the group G is a primitive permutation group with point stabilizer G_ω if and only if G_ω is maximal in G and contains no normal subgroup of G. M. Liebeck, C. Praeger and J. Saxl [LPS2] determined the factorizations $G = G_\omega B$, for G any almost simple group and B any maximal subgroup of G which does not contain $E(G)$. P. P. Palfy and J. Saxl extracted from these those factorizations $E(G) = G_\omega B$ with G_ω and B being of the same order, [PaSa]. See also [BP], where some general results on factorizations of almost simple groups are presented.

Let G_i be a group and π_i a permutation representation of G_i on the set Ω_i, for $i = 1, 2$. Then π_1 and π_2 are said to be *isomorphic*, if there exists an isomorphism $\phi : G_1 \to G_2$ and a bijection $\theta : \Omega_1 \to \Omega_2$ such that

$$g^{\pi_1}\theta = \theta g^{\phi\pi_2} \text{ for all } g \in G_1.$$

Two permutation groups are *permutation equivalent*, if their permutation representations are isomorphic. Our results read as follows.

Theorem 1.1 *Let G be a primitive permutation group on a set Ω of size m. Suppose that $G = G_\omega G_\omega^\alpha$ for G_ω a point stabilizer and for $\alpha \in \mathrm{Aut}(G)$. Then one of the following holds. (Conversely each of the listed groups gives rise to such an example.)*

(i) *G is affine:*
$G = E_{2^3}L_3(2) \wr X$, *where X is a transitive subgroup of S_n and $m = 2^{3n}$.*

(ii) *G is almost simple:*

(a) *$E(G) \cong P\Omega_8^+(q)$ and G is an extension of $E(G)$ by field automorphisms. $E(G)_\omega \cong P\Omega_7(q)$, $E(G)_\omega \cap E(G)_\omega^\alpha \cong G_2(q)$ and $m = q^4(q^4-1)$;*

(b) *$E(G) \cong Sp_4(q)$, $q > 2$ even, and G is an extension of $E(G)$ by field automorphisms. $E(G)_\omega \cong L_2(q^2)2$, $E(G)_\omega \cap E(G)_\omega^\alpha \cong \mathrm{Frob}(q^2+1 : 4)$ and $m = q^2(q^2-1)/2$;*

(c) *$G \cong Sp_4(2) \cong S_6$, $G_\omega \cong P\Gamma L_2(4) \cong S_5$, $G_\omega \cap G_\omega^\alpha \cong \mathrm{Frob}(5:4)$ and $m = 6$;*

(d) *$G \cong Sp_4(2)' \cong A_6$, $G_\omega \cong L_2(4) \cong A_5$, $G_\omega \cap G_\omega^\alpha \cong \mathrm{Frob}(5:2)$ and $m = 6$;*

(e) *$G \cong M_{12}$, $G_\omega \cong M_{11}$, $G_\omega \cap G_\omega^\alpha \cong L_2(11)$ and $m = 12$;*

(iii) *G is permutation equivalent to an blow–up K of index n of an almost simple group A: $K \leq A \wr S_n$, where A is a permutation group listed in (ii) and $m = |A : A_\omega|^n$. Moreover $E(G)_\omega = E(A)_\omega^n$ and there exists $a \in \mathrm{Aut}(A)$ such that $A = A_\omega A_\omega^a$ and such that $E(G)_\omega \cap E(G)_\omega^\alpha \cong (E(A)_\omega \cap E(A)_\omega^a)^n$.*

Remark. Let G be a primitive permutation group such that $G = G_\omega G_\omega^\alpha$ so that G is listed in Theorem 1.1. Note that it is possible by inspection of each case to determine those G for which it is possible to choose α such that $G = G_\omega G_\omega^\alpha$ and such that $\alpha^2 \in G$. In fact this can be done in all cases except those satisfying (ii)(b) or (iii) with $E(G) \cong (Sp_4(q))^n$ with q even and $q > 2$ for some positive integer n: indeed, for such G a suitable α exists if and only if $G \cap Aut((Sp_4(q))^n)$ contains L^n where L satisfies $Sp_4(q) \leq L \leq Aut(Sp_4(q))$ and $L/Sp_4(q)$ is a Sylow 2-subgroup of the group of field automorphisms of $Sp_4(q)$.

An equivalent formulation of Theorem 1.1 is