Some Existentially Closed Graphs

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Abstract

We give a characterization of the existentially closed models and pregeneric models of the theory of planar graphs and graphs without $K_{3,3}$-minor. We show that there are $2^{\aleph_0}$ countable existentially closed models in these classes and that all are elementary equivalent, though there are no model-companions.

1 Introduction

We deal with planar graphs and related classes. Since planarity is a notion from topology which is not first order, we say -according to the theorem of Kuratowski [1]- a graph is planar if it contains no subdivision of K_5 or $K_{3,3}$. It is then easy to show that the theory of planar graphs is axiomatizable but not finitely axiomatizable and that the class of planar graphs is an inductive class. We then characterize the pregeneric elements which are the star-closed graphs. A planar graph is star-closed if it is not possible to add a star, three edges with a common vertex, without leaving the class of planar graphs.

In contrast to the whole class of planar graphs the class of star-closed planar graphs has the amalgamation property AP in a natural way. The existentially closed planar graphs are the star-closed planar graphs which are finitely universal over pseudo-edges. A pseudo-edge is just a pair of vertices which could be joined by an edge leaving the graph planar. Finitely universal over pseudo-edges means that every finite planar graph is embeddable over every pseudo-edge. This describes the class of existentially closed planar graph. The existentially closed graphs without $K_{3,3}$-minor behave very similar but there are differences concerning the pregeneric ones.

The theory of existentially closed planar graphs is not the model-companion of the theory of planar graphs, because it is not axiomatizable. Existentially closed planar graphs are connected. This fact can not be described in a first order language. We will show that there exist $2^{\aleph_0}$ non isomorphic countable existentially closed planar graphs. Though there is no model-companion these graphs are all

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elementary equivalent. In [4] this was shown for a special subclass using an Ehrenfeucht-Fraïssé game. Later Reker [7] gave a proof for all existentially closed planar graphs by extending the previous technique. We give a much shorter proof for the whole class. This paper is a condensed and extented version of a part of [8]. The last section is included to give a feeling for what will happen for classes of graphs embeddable in other surfaces.

2 Planar Graphs and Graphs without $K_{3,3}$-Minor

We start with a remark on our use of *planar*. Planarity is a notion from topology which has a graph-theoretical characterization by the famous theorem of Kuratowski [1]. He considered finite graphs only, but his theorem carries over to the countable case, as was shown by Dirac and Schuster [2]. Unfortunately this is not true for the power of the continuum and of course not for even greater cardinalities. So some cardinality-considerations are necessary which are not expressible in the first order language of graph-theory. The theory of planar graphs is not first order. Nevertheless, we call a graph planar if it meets the conditions in Kuratowski's theorem (rather then to restrict ourselves to the countable case). The theory of those graphs is first order.

We identify a graph (G, E) with a structure $< G, R >$, where R is a binary, irreflexive and symmetric relation on G. A *subgraph* of a given graph G is a submodel of $< G, E >$, it is always an induced subgraph. A not necessary induced subgraph of a given graph G is called a *part* of G.

The language of our theory of planar graphs has exactly one binary relation symbol R. The axioms are as follows:
R is an irreflexive and symmetric relation and *There is no part homeomorphic to K_5 and $K_{3,3}$.*
The latter is expressible by an infinite set of sentences each requiering that a certain finite graph homeomorphic to K_5 or $K_{3,3}$ is not a part of the model of that theory.
The class of planar graphs, of models of the above theory, is denoted by $\mathcal{P}_0$. Another theory of graphs we are interested in, is the theory of graphs without $K_{3,3}$-minor. The second axiom is changed to *There is no part homeomorphic to $K_{3,3}$*. The class of models of this theory, is denoted by $\mathcal{P}_1$.

We study these two theories simultaniously to see which role the forbidden subdivisions of K_5 are playing in the theory of planar graphs.

Now we prove:

Theorem 2.1 The theory of $\mathcal{P}_i$ is not finitely axiomatizable for $i = 0, 1$.

Proof: Let G_0 be isomorphic to $K_{3,3}$. Fix an edge and let G_j be the graph where this edge is subdivided by exactly i vertices. For all natural numbers j G_j is not