Hadamard functions preserving nonnegative H-matrices

Ludwig Elsner
Fakultät für Mathematik,
Universität Bielefeld,
Postfach 100131, D-33501 Bielefeld, Germany

Daniel Hershkowitz*
Department of Mathematics
Technion - Israel Institute of Technology
Haifa 32000, Israel

Version dated: March 6, 1997
Printed: May 20, 1997

ABSTRACT

For k nonnegative $n \times n$ matrices $A_l = (a_{ij}^l)$ and a function

$$f : \mathbf{R}_+^k \to \mathbf{R}_+$$

consider the matrix

$$C = f(A_1, \ldots, A_k) = (c_{ij}),$$

where

$$c_{ij} = f(a_{ij}^1, \ldots, a_{ij}^k), \qquad i, j = 1, \ldots, n.$$

Denote by $\rho(A)$ the spectral radius of a nonnegative square matrix A, and by $\sigma(A)$ the minimal real eigenvalue of its comparison matrix $M(A) = 2\mathrm{diag}(a_{ii}) - A$. It is known that the function

$$f(x_1, \ldots, x_k) = c x_1^{\alpha_1} \ldots x_k^{\alpha_k}$$

where $\alpha_i \in \mathbf{R}_+, \quad \Sigma_{i=1}^k \alpha_i \geq 1$ and $c > 0$, satisfies the inequalities

$$\rho(f(A_1, \ldots, A_k)) \leq f(\rho(A_1), \ldots, \rho(A_k)),$$

*supported by SFB 343 "Diskrete Strukturen in der Mathematik", Bielefeld

as well as the inequalities

$$\sigma(f(A_1,\ldots,A_k)) \geq f(\sigma(A_1),\ldots,\sigma(A_k)).$$

whenever A_i are nonnegative H-matrices, i.e. $\sigma(A_i) \geq 0$. The last inequality implies that the above function f maps the set of nonnegative H-matrices into itself. In this note it is proven that these are the only continuous functions with this property.

---

1 Introduction

Let $\mathbf{R}_+$ be the set of nonnegative real numbers. For k nonnegative $n \times n$ matrices $A_l = (a_{ij}^l), l = 1,\ldots,k$ and a function

$$f : \mathbf{R}_+^k \to \mathbf{R}_+$$

we consider the matrix

$$f(A_1,\ldots,A_k) = C = (c_{ij}),$$

where

$$c_{ij} = f(a_{ij}^1,\ldots,a_{ij}^k), \qquad i,j = 1,\ldots,n.$$

Such matrix functions are called Hadamard functions, e.g. [4, Section 6.3].

Denote by $\rho(A)$ the spectral radius of a nonnegative square matrix A, and by $\sigma(A)$ the minimal real eigenvalue of its comparison matrix $M(A) = 2\mathrm{diag}(a_{ii}) - A$. Denote by HP_n the set of all n-by-n nonnegative H-matrices, i.e. the nonnegative matrices A for which $\sigma(A) \geq 0$. These are the nonnegative matrices whose comparison matrices are M-matrices.

Let K be a set of nonnegative matrices. We say that f preserves K, if $A_1,\ldots,A_k \in K$ implies $f(A_1,\ldots,A_k) \in K$. In this note we mainly study the functions f preserving HP_n. First, we show that for $n \geq 3$ such a function satisfies the inequality

$$\rho(f(A_1,\ldots,A_k)) \leq f(\rho(A_1),\ldots,\rho(A_k)) \tag{1.1}$$

for all nonnegative $n \times n$ matrices $A_i, i = 1,\ldots,k$. We then characterize all *continuous* functions preserving nonnegative H-matrices. It turns out that these are exactly the functions f of the form

$$f(x_1,\ldots,x_k) = cx_1^{\alpha_1}\ldots x_k^{\alpha_k} \tag{1.2}$$

where $\alpha_i \geq 0, c \geq 0$ and $\alpha = \Sigma_{i=1}^k \alpha_i \geq 1$. (The latter condition is required whenever $n > 2$.) These functions are also characterized by the inequality

$$\sigma(f(A_1,\ldots,A_k)) \geq f(\sigma(A_1),\ldots,\sigma(A_k)), \tag{1.3}$$

for all $A_i \in HP_n$ for some $n,\ n > 2$.

Note that while the functions (1.2) are characterized by (1.3), they are not the only functions satisfying (1.1). The set of those is charcterized in [2] by simple functional inequalities and it contains more functions, e.g. the function $f(x) = \min(x, x^2)$ for $k = 1$.