On uniqueness of invariant measures for finite and infinite dimensional diffusions

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Abstract

We prove uniqueness of "invariant measures", i.e., solutions to the equation $L^*\mu = 0$ where $L = \Delta + B \cdot \nabla$ on $\mathbb{R}^n$ with B satisfying some mild integrability conditions and μ is a probability measure on $\mathbb{R}^n$. This solves an open problem posed by S.R.S. Varadhan in 1980. The same conditions are shown to imply that the closure of L on $L^1(\mu)$ generates a strongly continuous semigroup having μ as its unique invariant measure. The question whether an extension of L generates a strongly continuous semigroup on $L^1(\mu)$ and whether such an extension is unique is addressed separately and answered positively under even weaker local integrability conditions on B. The special case when B is a gradient of a function (i.e., the "symmetric case") is in particular studied and conditions are identified ensuring that $L^*\mu = 0$ implies that L is symmetric on $L^2(\mu)$ resp. $L^*\mu = 0$ has a unique solution. We also prove infinite dimensional analogues of the latter two results and a new elliptic regularity theorem for invariant measures in infinite dimensions.

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1 Introduction and main results

This paper is concerned with the problem which was raised by S.R.S. Varadhan in [31] of uniqueness of invariant measures for diffusions with singular drifts on a finite or infinite dimensional state space X. More precisely, if $X = \mathbb{R}^n$, these diffusions are weak solutions of stochastic differential equations of type

$$dX_t = dW_t + B(X_t)dt \tag{1.1}$$

where $(W_t)_{t\geq 0}$ denotes Brownian motion on $\mathbb{R}^n$ and $B : \mathbb{R}^n \to \mathbb{R}^n$ is merely Borel-measurable. We emphasize, however, that many of our results also hold in case a smooth strictly positive definite diffusion matrix $\sigma(X_t)$ replaces the identity in front of dW_t in (1.1).

Since we do not want to impose any a-priori conditions that ensure the existence of a solution to (1.1) resp. a corresponding semigroup describing its transition probabilities, we formulate our problem purely analytically, i.e., in terms of the underlying generator $\Delta + B\cdot\nabla$, where Δ denotes the Laplacian on $\mathbb{R}^n$. Let us do so precisely, first sticking to the finite dimensional case $X = \mathbb{R}^n$.

For a positive Radon measure μ on a topological space X let $L^p(\mu)$ (($:= L^p(X;\mu)$) resp. $L^p_{loc}(\mu)$ (($:= L^p_{loc}(X;\mu)$), $p \geq 1$, denote the corresponding spaces of p-integrable resp. locally p-integrable real-valued functions on X. As usual for $p \geq 1$, $r > 0$, $H^{p,r}(\mathbb{R}^n)$ denotes the Sobolev space of all functions on $\mathbb{R}^n$ with generalized derivatives up to order r in $L^p(dx)$, where dx denotes Lebesgue measure on $\mathbb{R}^n$. $H^{p,r}_{loc}(\mathbb{R}^n)$ denotes the corresponding local Sobolev space, i.e., $f \in H^{p,r}_{loc}(\mathbb{R}^n)$ if $\zeta f \in H^{p,r}(\mathbb{R}^n)$ for all $\zeta \in C_0^\infty(\mathbb{R}^n)$.

Set

$$Lf := \Delta f + \langle B, \nabla f\rangle \quad , \ f \in C_0^\infty(\mathbb{R}^n), \tag{1.2}$$