Topological Invariance of Chow Varieties

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Abstract

By definition Chow varieties are invariants of embeddings of algebraic varieties in projective spaces. Hoyt showed that the algebraic homeomorphism type is determined by the algebraic isomorphism class of the variety. In this paper a rectified version of his proof is given. The use of resultants reduces to the problem, whether the (ring theoretic) greatest common devisor of a set of polynomials can be constructed via an algebraic map. The argument works for arbitrary characteristic. As an appendix we give a short description of the construction of Chow varieties in characteristic $p \neq 0$.

Introduction

In [6] Chow varieties of complex projective spaces $\mathbb{P}^n_{\mathbb{C}}$ are defined using resultants. The resultant of a variety in $\mathbb{P}^n_{\mathbb{C}}$ is a principal ideal in a polynomial ring over $\mathbb{C}$ which is generated by the Chowform of the variety. This form is unique up to a factor in $\mathbb{C}^*$. Hence its coefficients define a point in $\mathbb{P}^N_{\mathbb{C}}$, where N only depends on the dimension, codimension, and degree of V. By multiplication of Chow forms one gets a definition for Chow forms of formal sums of algebraic varieties of fixed dimension. Chow forms of arbitrary algebraic cycles are defined by linear extension. Fixing dimension r, codimension $n-r$, and degree d of cycles the coefficients of the Chow forms of the cycles build an algebraic set, which is called Chow variety $\mathcal{C}_{r,d}(\mathbb{P}^n_{\mathbb{C}})$.

The advantage of viewing Chow forms as generators of resultants is, that intersection of a variety with a hyperplane is given on Chow forms by evaluating the Chowform on the coefficients of the defining equation of the hyperplane.

Chow varieties of algebraic subsets $X \subset \mathbb{P}^n_{\mathbb{C}}$ are defined by restriction to cycles with support on X. Defining Chow varieties of arbitrary projective algebraic sets X now means choosing an embedding $i : X \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ and taking the Chow variety $\mathcal{C}_{r,d}(X,i)$ of $i(X)$.

For the definition of Lawson homology one needs the homotopy type independent of the embedding i. Hoyt [4] proved the homeomorphism type of $\coprod_{d\geq 0} \mathcal{C}_{r,d}(X,i)$ to be independent of embeddings. The purpose of this note is to clarify and straighten out Hoyt's approach for arbitrary fields. In fact by Barlet [1] one knows that $\mathcal{C}_{r,d}(X,i)$ is independent of i up to isomorphism of algebraic sets in case of $K=\mathbb{C}$. On the other hand Nagata [5] shows that the Chow variety of 0-dimensional cycles of degree 2 in $\mathbb{P}^3_{\mathbb{Z}_2}$ is not normal, but there is an embedding into some $\mathbb{P}^n_{\mathbb{Z}_2}$ such that the Chow variety of the embedded $\mathbb{P}^3_{\mathbb{Z}_2}$ becomes normal. Hence those two Chow varieties can not be isomorphic algebraic sets.

As an appendix we describe the construction of Chow forms over arbitrary fields in section 3. We will explain how the Chow variety can be defined in case of inseparability. For the same reasons the Chowform of a variety is the generator of the resultant as in the case $K=\mathbb{C}$.

In the first section we will reduce the proof of Hoyt's theorem to the problem of determining the greatest common divisor of a set of multihomogeneous polynomials. The algebraicity of the greatest common divisor in the coefficients of the multihomogeneous polynomials is shown in section 2.

We will consider projective algebraic sets whose vanishing ideals are defined over an arbitrary field K, morphisms of algebraic sets are K-morphisms. The zero sets of such an homogeneous ideal are viewed as subsets of $\mathbb{P}^n_\Omega$, where Ω is an algebraically closed field extension of K of countably infinite transcendence degree over K. Irreducible algebraic sets are called varieties. In this sense the name Chow variety may be a bit confusing, because in general Chow varieties are reducible.

Points in $\mathbb{P}^n$ are denoted $\underline{p} := (p_0,\ldots,p_n)$. For a field K we denote by $K[\underline{p}]$ the ring $K[p_0,\ldots,p_n]$, where the representative $(p_0,\ldots,p_n)$ is chosen such that all non vanishing p_i are transcendental over K. $K(\underline{p})$ is the quotient field of $K[\underline{p}]$.

1 Independence of Chow varieties of embeddings

Definition 1.1. Let X be a projective variety (over K) with a (K-)embedding $j: X \hookrightarrow \mathbb{P}^n_\Omega$. Denote by $\mathcal{C}_{r,d}(X,j)$ the Chow variety of cycles V, such that $\deg V = d$ and $supp(V) \subset j(X)$. Further let $\mathcal{C}_r(X,j) := \coprod_{d\geq 0} \mathcal{C}_{r,d}(X,j)$.

Generally we know that Chow varieties of projective varieties depend on embeddings [5]. But in the case of ground field $\mathbb{C}$ the independence of the chosen embedding arises from analytic arguments as we will see below. In the case of characteristic $p \neq 0$ the algebraic sets can be topologized using