Spectral Theory of Some Matrix Differential Operators of Mixed Order

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1. Introduction

Our purpose in this work is to study the spectral properties and the scattering theory of self-adjoint operator matrices of the form

$$H = \begin{pmatrix} -\Delta + q & -\nabla b + v \\ \bar{b}\mathrm{div} + v^* & h \end{pmatrix} \tag{1}$$

acting in the product of Hilbert spaces $\mathcal{H} = L_2(\mathbb{R}^n;\mathbb{C}^n)\times L_2(\mathbb{R}^n),\ n \geq 1$. Here $L_2(\mathbb{R}^n;\mathbb{C}^n)$ is L_2-space of vector functions $u : \mathbb{R}^n \to \mathbb{C}^n,\ q$ is an $n\times n$ Hermitian matrix function, b is a complex function, v is an $n \times 1$ matrix function, h is a real function. In the recent paper [AtLMS] (see also [S]) an abstract approach was proposed to study the essential spectrum of 2×2 block operator matrices acting in a Banach space. In particular, this approach was proved to be very useful for dealing with the spectral theory of some mixed order differential operators occuring in magnetohydrodynamics (see [AtLMS], [FMM], [LM]). On the other hand the structure of the continuous spectrum of such matrix differential operators has not been investigated. Here we give a sufficiently complete analysis of the self-adjoint operators of the form (1). In particular, we prove (under certain assumptions) the limiting absorption principle and the absence of singular continuous spectrum. Moreover, we describe the possible points of the eigenvalue accumulation. In Section 2 we study the essential spectrum of some general self-adjoint operator matrices of the form

$$L = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}. \tag{2}$$

In particular, we prove (cf. [S]) that under some natural assumptions $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0)$, where

$$H_0 = \begin{pmatrix} -\Delta & 0 \\ 0 & h_0 \end{pmatrix}, \tag{3}$$

$$h_0 = h - |b|^2 \tag{4}$$

In Section 3 we present some variant of Mourre's commutator method, which is suitable for dealing with the operators of the form (1). The limiting absorption principle for the operator H is proved in Section 4. In Section 5 we show that for the pair H, H_0 there exist complete wave operators. Note that the case $b(x) \equiv 0$ was investigated in [K].

2. The essential spectrum of some matrix operators.

First introduce some notations. Let T be a linear closed operator in a Hilbert space with the domain $D(T)$. Denote by $\rho(T)$ the resolvent set of T, $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is the spectrum of T. The set $\sigma_{\mathrm{ess}}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is not a Fredholm operator}\}$ is called the essential spectrum of T. If T is a self-adjoint operator, then $\sigma_{\mathrm{ess}}(T) = \{\lambda \in \sigma(T) \mid \lambda \text{ is not an isolated eigenvalue of finite multiplicity}\}$. Denote by $\mathcal{L}(\mathcal{H}, \mathcal{K})$ ($S_\infty(\mathcal{H}, \mathcal{K})$) the space of bounded (respectively compact) linear operators from $\mathcal{H}$ into $\mathcal{K}$. $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$, $S_\infty(\mathcal{H}) = S_\infty(\mathcal{H}, \mathcal{H})$.

Consider symmetric operator matrices of the form (2) acting in the product of Hilbert spaces $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$. Suppose that A and C are self-adjoint operators in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Let B be a densely defined closable operator from $\mathcal{H}_2$ into $\mathcal{H}_1$ such that $D(B)$ is a core of C. We shall assume that

$$D(B^*) \supset D(|A|^{1/2}). \tag{5}$$

Under these conditions L is essentially self-adjoint on $D(A) \times D(B)$ (see [ALMS], [MS]). We retain the notation L for the closure of the operator (2). Define

$$S_0(\lambda) := B^*(A - \lambda)^{-1}B, \qquad \lambda \in \rho(A).$$

By (5), the closure $S := \overline{S_0(\lambda)}$ is a bounded operator in $\mathcal{H}_2$. Define on $D(C)$

$$T(\lambda) = C - S(\lambda), \qquad \lambda \in \rho(A). \tag{6}$$

Note that $\lambda \in \rho(T(\lambda))$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ [AL].

Define

$$F(\lambda) := B^*(A - \lambda)^{-1} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2), \qquad \lambda \in \rho(A).$$

Then for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$(L - \lambda)^{-1} = \begin{pmatrix} (A - \lambda)^{-1} + (F(\bar\lambda))^*(T(\lambda) - \lambda)^{-1}F(\lambda) & -(F(\bar\lambda))^*(T(\lambda) - \lambda)^{-1} \\ -(T(\lambda) - \lambda)^{-1}F(\lambda) & (T(\lambda) - \lambda)^{-1} \end{pmatrix} \tag{7}$$

(see [AL], [ALMS], [MS]). Denote

$$L(\mu) := \begin{pmatrix} A & 0 \\ 0 & T(\mu) \end{pmatrix}, \qquad \mu \in \rho(A).$$

Theorem 2.1. *Suppose that for some (and hence for all) $\lambda \in \mathbb{C} \setminus \mathbb{R}$*

$$G(\lambda) := (C - \lambda)^{-1}B^*(A - \lambda)^{-1} \in S_\infty(\mathcal{H}_1, \mathcal{H}_2). \tag{8}$$