CYCLICALLY PRESENTED GROUPS AND TAKAHASHI MANIFOLDS AS CYCLIC COVERINGS

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Abstract

We consider groups with cyclic presentations which arise as fundamental groups of a family of closed 3-dimensional manifolds. These manifolds were firstly described by Takahashi using Dehn surgeries on links. We demonstrate that the cyclic automorphisms of groups induce cyclic coverings of the 3-sphere branched over 2-bridge knots. Moreover, polynomials associated with cyclic presentations are equal to Alexander polynomials of corresponding knots.
Keywords: fundamental group, 3-manifold, cyclic covering.

1. Cyclically Presented Groups

The cyclically presented groups comprise a rich source of groups which are interesting from a topological point of views. The connection between cyclically presented groups and cyclic branched coverings of knots and links was studied, in particular, in [2], [5], [6] and [7].

Let $F_n = \langle x_1, \dots, x_n \quad | \quad \rangle$ be the free group of rank n and $\eta : F_n \to F_n$ be an automorphism of order n such that $\eta(x_i) = x_{i+1}$ for $i = 1, \dots, n$, where all indices by mod n. We recall [9, §9] that for a reduced word $w \in F_n$ the *cyclically presented group* $G_n(w)$ is given by

$$G_n(w) = \langle x_1, \dots, x_n \quad | \quad w, \eta(w), \dots, \eta^{n-1}(w) \rangle.$$

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A group G is said to *have a cyclic presentation* if $G \cong G_n(w)$ for some n and w. Clearly, the automorphism η of F_n induces an automorphism of $G_n(w)$. Such automorphism defines an action of the group $\mathbb{Z}_n = \langle \eta \mid \eta^n = 1 \rangle$ on $G_n(w)$. Let us consider a split extension $H_n = G_n(w) \lambda \mathbb{Z}_n$. The group H_n is said to be a *natural extension of a cyclically presented group.* It was remarked in [5] that the group $H_n = H_n(v)$ always has a 2-generator, 2-relator presentation of the form

$$H_n(v) \cong \langle \eta, x \quad | \quad \eta^n = v(\eta, x) = 1 \rangle,$$

where $v = v(\eta, x) = w(x, \eta^{-1} x \eta, \ldots, \eta^{-(n-1)} x \eta^{n-1})$.

Following [9], we define the *polynomial* $f_w(t)$ *associated with* the cyclically presented group $G \cong G_n(w)$ as

$$f_w(t) = \sum \alpha_i t^i,$$

where α_i is the exponent sum of x_i in w, $1 \leq i \leq n$.

We recall that in [13] L. Neuwirth describes an algorithm for deciding is a group presentation with n generators and n relations corresponds to the spine of a closed compact 3-manifold. The nice approach to determine which cyclically presented groups arise as fundamental groups of 3-manifolds was devoloped by M. Dunwoody in [5], where he described an algorithm to enumerate Heegaard diagrams with the necessary cyclic symmetry. In particularly, he asked about covering properties of manifolds which have been arised in this way.

In the present paper we give another approach to describe a series of cyclically presented groups which arise as fundamental groups of 3-manifolds. We consider Dehn surgeries of a $2n$-component chain with cyclically symmetric surgery parameters. We demonstrate that this symmetry corresponds a cyclic coverings of two-bridge knots (Theorem 4.1 and 4.2) and descibe correspondong cyclically presented groups (Theorem 5.1). In particularly, Dunwoody's groups with $w = w(x_i, x_{i+1}, x_{i+2})$ belong to this series. We will start from two well-known examples.