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# Optimal Superimposed Codes and Designs for Renyi's Search Model

*Abstract–* **We consider a class of superimposed codes [1, 2] for the search model of A. Renyi [3]. A lower bound on the code length is obtained and optimal superimposed codes are presented.**

## 1. Notations and definitions.

Let $1 \leq s < k < t$, $N > 1$ be integers and $X = \|x_i(u)\|$, $i = 1, 2, \ldots, N$, $u = 1, 2, \ldots, t$ be a binary $N \times t$–matrix (code) with columns (codewords) $\mathbf{x}(1), \mathbf{x}(2), \ldots, \mathbf{x}(t)$, where $\mathbf{x}(u) = (x_1(u), x_2(u), \ldots, x_N(u))$. We say that the binary column $\mathbf{x}$ *covers* the binary column $\mathbf{y}$ if the Boolean sum $\mathbf{x} \vee \mathbf{y} = \mathbf{x}$. The code $X$ is called [1, 2] a *superimposed* $(s, t)$*-code* if the Boolean sum of any $s$-subset of columns of $X$ covers those and only those columns of $X$ which are the terms of the given Boolean sum. The code $X$ is called [1, 2] a *superimposed* $(s, t)$*-design* if all Boolean sums composed of not more than $s$ columns of $X$ are distinct.

Let $\mathbf{x}_i = (x_i(1), x_i(2), \ldots, x_i(t))$, $i = 1, 2, \ldots, N$, be the i-th row of $X$ and

$$|\mathbf{x}_i| = \sum_{u=1}^{t} x_i(u), \quad i = 1, 2, \ldots, N, \quad 0 \leq |\mathbf{x}_i| \leq t,$$

be the number of 1-entries (weight) in row $\mathbf{x}_i$.

**Definition 1**. An $N \times t$ matrix $X$ is called a superimposed $(s, t, k)$-code (design) of *length* $N$*, size* $t$*, strength* $s$ and *constraint* $k$ if two conditions are fulfilled:

- code $X$ is a superimposed $(s, t)$-code (design),
- for any $i = 1, 2, \ldots, N$, weight $|\mathbf{x}_i| \leq k$.

**Remark 1**. The consraint on the weight of rows $|\mathbf{x}_i| \leq k$, $i = 1, 2, \ldots, N$, was introduced by A. Renyi [3] and was studied by G. Katona [4] for the search designs.

## 2. Lower bound

**Proposition 1**. *Let $k \geq s \geq 2$. For any superimposed $(s-1, t, k)$-code $((s, t, k) - design)$ $X$ of length $N$, the following inequality takes place*

$$N \geq \left\lceil \frac{st}{k} \right\rceil, \tag{1}$$

*where $\lceil a \rceil$ denotes the least integer $\geq a$. In addition, if $t = kq$, $q = 2, 3, \ldots$ and there exists superimposed $(s-1, kq, k)$-code $((s, kq, k) - design)$ $X$ of length $N = sq$, then $X$ is a constant weight code of weight $s$ and for any $i = 1, 2, \ldots, N$, the weight of row $|\mathbf{x}_i| = k$.*

**Proof**. Consider an arbitrary superimposed $(s-1, t, k)$-code $X$ of length $N$. Let $n, 0 \leq n \leq t$, be the number of codewords of $X$ having a weight $\leq s-1$. From definition of superimposed $(s-1, t)$-code it follows (see, [1]) that $n \leq N$ and, for each codeword of weight $\leq s-1$, there exists a row in which all the remaining elements, except for the element of this codeword, are 0's. We delete these $n$ rows from $X$ together with $n$ codewords of weight $\leq s-1$. Consider the remaining $(N-n) \times (t-n)$ matrix $X'$. Obviously, each column of $X'$ has a weight $\geq s$ and each its row contains $\leq k$ 1's. Hence,

$$s(t-n) \leq k(N-n). \tag{2}$$

It follows $ts \leq kN - n(k-s) \leq kN$, provided that $k \geq s$. For superimposed $(s-1, t, k)$-code, inequality (1) is proved. Let $q = 2, 3, \ldots$ be integer and $t = kq$. It is easy to see that both of inequalities (1) and (2) have signs of equality, iff superimposed $(s-1, kq, k)$-code $X$ is a constant weight code of weight $s$ and for any $i = 1, 2, \ldots, N$, the weight of row $|\mathbf{x}_i| = k$. For superimposed $(s-1, t, k)$-code, Proposition 1 is proved.

It is known [2] that code $X$ is superimposed $(s, t, k)$-design iff $X$ is superimposed $(s-1, t, k)$-code and all $\binom{t}{s}$ Bolean sums composed of $s$ columns of $X$ are distinct. Hence, inequality (1) is true for superimposed $(s, t, k)$-design.

Proposition 1 is proved.

Denote by $N(s, t, k)$, $(\tilde{N}(s, t, k))$ the minimal possible length of superimposed $(s, t, k)$-code ($(s, t, k)$-design). From Proposition 1 it follows that for $k \leq s$,

$$N(s-1, t, k) = \tilde{N}(s, t, k) = t$$

and for $k \geq s+1$,

$$\tilde{N}(s, t, k) \geq N(s-1, t, k) \geq \left\lceil \frac{st}{k} \right\rceil.$$