Immediate extensions of rings and approximation of roots

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Dedicated to Professor Irving Kaplansky on his 80^{th} birthday

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Abstract

In this work we generalize the notion of immediate extensions of valued fields introduced in Krull's work [21] to arbitrary commutative rings. We show that every ring possesses maximal immediate extensions and point out the importance of linear compactness in the investigation of maximal immediate extensions of local rings as one of the natural classes of maximally complete rings. Another important, in some sense exceptional class of maximally complete rings is that of von Neumann regular rings. A necessary and sufficient condition is given for the linear compactness of any maximal immediate extension of a valuation ring. Newton's Approximation Method is used to give a short direct proof of Hensel's Lemma and to extend a result of Lang [22] for finding simple roots of polynomials over linearly compact local rings. A somewhat weaker version of Lang's condition for the existence of simple roots whose congruence classes are multiple roots modulo the maximal ideal is given by using an idea of Kaplansky. Results of Teichmüller, Kaplansky and Cohen on the existence of coefficient fields are also generalized to local linearly compact rings. In the last section Kaplansky's method [14], as a non-linear version of Newton Approximation, is used to find (possibly multiple) roots of polynomials, and this leads to an extension of Mac Lane's characterization of algebraically closed linearly compact valued fields to linearly compact valuation rings. Moreover, Kaplansky's result on the uniqueness of maximal immediate extensions of valued fields is reformulated as a result on the location and the approximation of roots, the classical topic for real polynomials. Therefore this reformulation yields new aspects of Kaplansky's result and points out at the same time a natural analogy between linearly compact valued fields satisfying Hypothesis A and the field $\mathbb{R}$ of real numbers or more generally, complete real closed fields.

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1 Introduction

The origins of this work can be found in the works of Krull [21], Ostrowski [28] and Kaplansky [14] on maximal extensions of valued fields. It is well known that the value group and the residue field do not characterize a valued field. For example, for every field K of prime characteristic p and every totally ordered group Γ, there are three fields which have K as residue field and Γ as value group: F_1 of characteristic 0, F_2 and F_3 of characteristic p. Note that in contrast to the existence of F_2 and F_3, for which we can simply take the quotient field of the group algebra $K\Gamma$ and the much bigger field $K[[\Gamma]]$ of generalized power series, respectively, the existence of F_1 is highly non-trivial. For the construction of F_1 one can use Krull's "Aufspaltungsprozess" to reduce the problem to the case when Γ is archimedean or equivalently F_1 is of rank 1, and then construct F_1 with the help of Witt rings. This observation led probably to the creation of the theory of maximal valued fields in [21]: these are those fields which are maximal in the set of valued fields with given residue field and value group. The theory of maximal valued fields culminated in the work [14] of Kaplansky. Roughly speaking, Kaplansky's result states that one can get all maximally complete valued fields by using Krull's Aufspaltungsprozess and Witt rings in the unequal characteristic case and generalized power series fields in the equal characteristic case. Kaplansky discussed also the uniqueness problem of maximal extensions and reduced it to the existence problem of roots of certain polynomials. Then he solved this question almost completely by giving both sufficient conditions for the existence of roots and a transfinite approximation of roots which is in fact an extension or, more precisely, a non-linear version of the Newton Approximation for valued fields; this approximation is also suitable for finding multiple roots. We shall call this method Kaplansky's method. As was already observed by Kürschák, the inventor of valuation theory, valued fields are not generalizations of ordered (i.e., formally real) fields: their theory is in some sense paralell to that of the latter. Together with Hensel's Lemma, Kaplansky's method presents a way to locate roots of an equation over maximally complete valued fields, in parallel to the classical topic of location of real roots. For real polynomials the Newton Approximation gives a very effective way to approximate single roots. It may be less effective, however, for multiple roots. In the parallel theory, approximation of single roots in (maximally) complete fields can be done by the help of Hensel's Lemma. However, the Newton Approximation does not work for multiple roots in maximally complete valued fields and very little is known about their approximation. We will show in this work that Kaplansky's method provides a powerful direct method to approximate multiple roots in some maximally complete valued fields which can be considered as natural analogues of the field of real numbers or, more generally, complete real closed fields. Of course, there is an indirect handy way to use Newton's method after eliminating multiple roots by dividing out the greatest common divisor of a polynomial and its derivative for characteristic 0 or by reducing a polynomial to its separable form for prime characteristic.

In the present paper we shall extend the notion of immediate extension to arbitrary commutative rings and discuss the close relationship between linear compactness and maximal extensions of local rings. Our definition of immediate extension can be extended to all associative rings, but that will be done in a subsequent paper.

In the last two sections we discuss the approximation of (possibly multiple) roots by using the methods of Newton and Kaplansky. Some of our results are of a general character, i.e. they hold for all linearly compact local rings, or even for non-commutative rings. We cannot extend, however, our discussion of Kaplansky's method to local rings. We can do it only for linearly compact valuation rings or even for a more special class of linearly compact valuation rings, although we are convinced of the broader usability of this method.