Cyclotomic q–Schur algebras

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1 Introduction

Recently, we [**9**] and, independently, Du and Scott [**11**], defined an analogue of the q–Schur algebra [**6, 7**] for an Iwahori–Hecke algebra of type **B**. In this paper we study an analogue of the q–Schur algebra for an arbitrary Ariki–Koike algebra.

The Ariki–Koike algebra $\mathscr{H}$ is a cyclotomic algebra of type $G(r,1,n)$ [**2**], and it becomes the Iwahori–Hecke algebra of type **A** or **B** when $r=1$ or 2 respectively. By working over a ring R which contains a primitive rth root of unity, and by specializing the parameters appropriately, the Ariki–Koike algebra turns into the group algebra $R(C_r \wr \mathfrak{S}_n)$ of the wreath product of the cyclic group C_r of order r with the symmetric group $\mathfrak{S}_n$ of degree n.

For each multicomposition λ of n, we construct a right ideal M^λ of $\mathscr{H}$ (see Definition 3.8). The cyclotomic q–Schur algebra is then defined to be $\mathscr{S} = \operatorname{End}_{\mathscr{H}}(\oplus_\lambda M^\lambda)$. (Under the specialization above where $\mathscr{H} \cong R(C_r \wr \mathfrak{S}_n)$, the module M^λ becomes a module induced from a subgroup of the form $(C_r \times \cdots \times C_r) \rtimes \mathfrak{S}_\lambda$.)

A.M.S. subject classification (1991): 16G99, 20C20, 20G05
This paper is a contribution to the DFG project on "Algorithmic number theory and algebra". The authors acknowledge support from DFG; the third author was also supported in part by SERC grant GR/J37690 and by the Sonderforschungsbereich 343 in Bielefeld.

In this paper we construct a cellular basis for the cyclotomic q–Schur algebra. As a consequence we obtain a Weyl module W^λ for each multipartition λ of n. We show that W^λ has simple head F^λ and that the set $\{F^\lambda\}$, as λ ranges over the multipartitions of n, is a complete set of non–isomorphic irreducible $\mathscr{S}$–modules. Using the cellular structure of $\mathscr{S}$, it is now easy to see that the cyclotomic q–Schur algebra is quasi–hereditary.

In order to prove these results about the cyclotomic q–Schur algebra, we need to examine the ideals M^λ in some detail. Using the cellular structure of the Ariki–Koike algebra $\mathscr{H}$ (cf. [**10**, **13**]), we obtain a basis of M^λ and a special series of submodules of M^λ, known as a Specht series. From the Specht series of M^λ we construct the cellular basis of $\mathscr{S}$.

2 The Ariki–Koike algebra

Throughout this paper, r and n will be fixed positive integers with $r \geq 1$ and $n \geq 1$.

Let R be a commutative ring with 1 and let $q, Q_1, \dots, Q_r$ be elements of R with q invertible. The Ariki–Koike algebra $\mathscr{H}$ is the associative unital R–algebra with generators $T_0, T_1, \dots, T_{n-1}$ subject to the following relations

$$\begin{aligned}
(T_0 - Q_1)\cdots(T_0 - Q_r) &= 0 \\
T_0T_1T_0T_1 &= T_1T_0T_1T_0 \\
(T_i + 1)(T_i - q) &= 0 && \text{for } 1 \leq i \leq n-1 \\
T_{i+1}T_iT_{i+1} &= T_iT_{i+1}T_i && \text{for } 1 \leq i \leq n-2 \\
T_iT_j &= T_jT_i && \text{for } 0 \leq i < j-1 \leq n-2.
\end{aligned}$$

Suppose that $\pi \in R$ is a primitive rth root of 1 and that $q = 1$ and $Q_k = \pi^k$ for $k = 1, 2, \dots, r$. Then it follows from the definition that $\mathscr{H}$ is isomorphic to $R(C_r \wr \mathfrak{S}_n)$.

Let $\mathfrak{S}_n = \mathfrak{S}(\{1, 2, \dots, n\})$ be the symmetric group on $\{1, 2, \dots, n\}$ and let $s_i = (i, i+1)$ for $1 \leq i < n$. Then $s_1, s_2, \dots, s_{n-1}$ are the standard Coxeter generators of $\mathfrak{S}_n$. If $w \in \mathfrak{S}_n$ then a word $w = s_{i_1} \dots s_{i_k}$ for w is a reduced expression for w if k is minimal; in this case we say that w has length k and write $\ell(w) = k$. Given a reduced expression $s_{i_1} \dots s_{i_k}$ for $w \in \mathfrak{S}_n$, let $T_w = T_{i_1} \dots T_{i_k}$; the relations in $\mathscr{H}$ ensure that T_w is independent of the choice of reduced expression. We denote by $\mathscr{H}(\mathfrak{S}_n)$ the subalgebra of $\mathscr{H}$ generated by $T_1, T_2, \dots, T_{n-1}$. Then $\mathscr{H}(\mathfrak{S}_n)$ has basis $\{\, T_w \mid w \in \mathfrak{S}_n \,\}$, and it is isomorphic to an Iwahori–Hecke algebra of type **A**.

Define elements $L_m = q^{1-m} T_{m-1} \dots T_1 T_0 T_1 \dots T_{m-1}$ for $m = 1, 2, \dots, n$; these are analogues of the q–Murphy operators of the Iwahori–Hecke algebras of type **A** [**5**, **15**]. An easy calculation using the relations in $\mathscr{H}$ (cf. [**5**, (2.1), (2.2)]) shows that we have the following results.

(2.1) Suppose that $1 \leq i \leq n-1$ and $1 \leq m \leq n$. Then
(i) L_i and L_m commute.
(ii) T_i and L_m commute if $i \neq m-1, m$.
(iii) T_i commutes with L_iL_{i+1} and $L_i + L_{i+1}$.