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Murphy operators and the centre of the Iwahori–Hecke algebras of type **A**

Andrew Mathas*

Introduction

In [**4**] Murphy showed that for any field $\mathbb{F}$ the centre of the group algebra of the symmetric group $\mathfrak{S}_n$ on n symbols is the set of symmetric polynomials in the Murphy operators. One consequence of this result is a relatively easy proof of the Nakayama conjecture for $\mathbb{F}\mathfrak{S}_n$.

Given an invertible element q in a ring R let $\mathscr{H} = \mathscr{H}_{R,q}(\mathfrak{S}_n)$ be the associated Iwahori–Hecke algebra. Then $\mathscr{H}$ contains elements which are q–analogues of the Murphy operators of the symmetric group and once again the symmetric polynomials in these elements belong to the centre of $\mathscr{H}$. It is natural therefore to make the following conjecture.

(1.1) Conjecture (Dipper–James [2]) For any ring R and any invertible element q in R the centre of the Iwahori–Hecke algebra $\mathscr{H}_{R,q}(\mathfrak{S}_n)$ is the set of symmetric polynomials in the Murphy operators.

Dipper and James [**2**, Theorem 2.14] have proved this conjecture in the case where $\mathscr{H}$ is semisimple; unfortunately, there is a gap in their proof for the non–semisimple case. As for the symmetric group, one of the reasons why this conjecture is interesting is that as a corollary one can prove the Nakayama conjecture for $\mathscr{H}$ (this was proved by Gordon James and the author in [**3**], with one direction being done previously in [**2**]).

In this paper we reduce Conjecture 1.1 to a purely combinatorial problem of showing that certain polynomials are orthogonal. To the best of our knowledge these polynomials have not appeared in the literature previously; it seems likely that they will be of independent interest.

One of the reasons why the conjecture for $\mathscr{H}$ is more difficult to prove than in the symmetric group case is that the multiplication in $\mathscr{H}$ is much more complicated. In the first section of this paper we overcome this difficulty by proving a combinatorial result which allows us to rewrite an arbitrary product of Murphy operators as a linear combination of less complicated products (modulo "unimportant" terms). In the second section we apply this result towards Conjecture 1.1.

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2 Rewriting rules for Murphy operators

Throughout this paper we fix a positive integer n and let $\mathfrak{S}_n = \mathfrak{S}(\{1, 2, \dots, n\})$ be the symmetric group on $\{1, 2, \dots, n\}$.

Let R be a commutative ring with 1 and q an invertible element of R. Then the Iwahori–Hecke algebra $\mathscr{H} = \mathscr{H}_{R,q}(\mathfrak{S}_n)$ is the unital associative R–algebra with generators $T_1, T_2, \dots, T_{n-1}$ and relations

$$\begin{aligned} T_i^2 &= (q-1)T_i + q, \\ T_jT_{j+1}T_j &= T_{j+1}T_jT_{j+1}, \\ T_iT_j &= T_jT_i \quad \text{if } |i-j| > 2, \end{aligned}$$

for all $1 \leq j < i < n$.

Given an integer i, where $1 \leq i < n$, let $s_i = (i, i+1)$; then $\{s_1, s_2, \dots, s_{n-1}\}$ is the (standard) set of Coxeter generators for the symmetric group $\mathfrak{S}_n$. Suppose that $w \in \mathfrak{S}_n$ and write $w = s_{i_1}s_{i_2}\dots s_{i_k}$; this expression for w is reduced if k is minimal, in which case we say that w has length $\ell(w) = k$. Given such a reduced expression for w let $T_w = T_{i_1}T_{i_2}\dots T_{i_r}$; the relations in $\mathscr{H}$ ensure that T_w is independent of the choice of reduced expression for w. Moreover, $\{\, T_w \mid w \in \mathfrak{S}_n \,\}$ is a basis for $\mathscr{H}$.

(2.1) Definition [2] *The* Murphy operators *of $\mathscr{H}$ are the elements $L_1 = 0$ and*

$$L_i = q^{1-i}T_{(1,i)} + q^{2-i}T_{(2,i)} + \dots + q^{-1}T_{(i-1,i)},$$

for $i = 2, \dots, n$.

;sing the defining relations in $\mathscr{H}$ it is a straightforward albeit tedious exercise to show that:

(2.2) Suppose that $1 \leq i < n$ and $1 \leq j \leq n$. Then
- (i) T_i and L_j commute when $i \neq j-1, j$.
- (ii) T_i commutes with L_iL_{i+1} and $L_i + L_{i+1}$.
- (iii) L_i and L_j commute.

By (i) and (ii) the symmetric polynomials in the Murphy operators belong to the centre of $\mathscr{H}$.

Now that we have amassed sufficient notation to explain Conjecture 1.1 in the general case we restrict our attention to the generic Iwahori–Hecke algebra (where we renormalise everything).

(2.3) Let $q^{\frac{1}{2}}$ be an indeterminate over $\mathbb{Z}$ and let $\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ be the ring of Laurent polynomials in $q^{\frac{1}{2}}$ with integer coefficients. Then $\mathscr{H} = \mathscr{H}_{\mathcal{A},q}(\mathfrak{S}_n)$ is the generic Iwahori–Hecke algebra of $\mathfrak{S}_n$.