Reflections and almost concealed canonical algebras

Thomas Hübner

Abstract. Given a tilting sheaf $\mathcal{T}$ over a weighted projective line $\mathbb{X} = \mathbb{X}(\underline{\lambda}, \underline{\mathbf{p}})$ (cf. [**4**]), its endomorphism ring $\Sigma = \mathrm{End}(\mathcal{T})$ is (by definition) an almost concealed canonical algebra obtained by tilting at $\mathcal{T}$. The indecomposable direct summands T_i of $\mathcal{T}$ can be partitioned into *sources* and *sinks*, according as their corresponding source and sink maps (in add $\mathcal{T}$) are both monomorphisms or both epimorphisms (cf. Theorem 0.3). As a consequence, each almost complete tilting sheaf admits *exactly two* complements (Corollary 0.4). -

By Theorem 0.5, T_i is a source [sink, resp.] in add $\mathcal{T}$ if and only if the corresponding simple Σ-module S_i is torsion-free [torsion, resp.] with respect to the split torsion theory $(\mathrm{mod}_+ \Sigma, \mathrm{mod}_- \Sigma)$ of $\mathrm{mod}\,\Sigma$ obtained by tilting at $\mathcal{T}$. - The linking idea in achieving this result is the concept of *mutations*: roughly spoken, the position of simple Σ-modules within the derived category $\mathcal{D}^b\mathrm{coh}\,\mathbb{X}$ of $\mathrm{coh}\,\mathbb{X}$ is obtained by the "same kind" of mutations (cf. Proposition 3.1) which also allow the construction of sink/source maps for add $\mathcal{T}$ (Proposition 2.3).

Introduction

Throughout, let k denote an algebraically closed field. -

Let $\mathrm{coh}\,\mathbb{X}$ denote the category of coherent sheaves over a weighted projective line $\mathbb{X} = \mathbb{X}(\underline{\lambda}, \underline{\mathbf{p}})$ (in the sense of [**4**]) attached to a weight sequence $\underline{\mathbf{p}} = (p_1, \ldots, p_t)$ of integers $p_i \geq 1$ and a parameter sequence $\underline{\lambda} = (\lambda_1, \ldots, \lambda_t)$ of pairwise different points $\lambda_i \in \mathbb{P}^1(k)$. - The category $\mathrm{coh}\,\mathbb{X}$ is an abelian k-category, and hereditary since $\mathrm{Ext}^2(-,-) \equiv 0$; it has a tilting object $\mathcal{T}$ whose endomorphism ring is a canonical algebra Λ in the sense of [**12**]. Hence the Grothendieck group $\mathrm{K}_0\,\mathbb{X}$ is finitely generated free; it is endowed with a bilinear form - the *Euler characteristic* - induced by

$$\langle E, F\rangle \quad = \quad \dim_k \mathrm{Hom}(E,F) - \dim_k \mathrm{Ext}^1(E,F) \qquad \text{for} \quad E, F \in \mathrm{coh}\,\mathbb{X} \; .$$

Furthermore, $\mathrm{coh}\,\mathbb{X}$ is $\mathbb{L}(\underline{\mathbf{p}})$-graded, where $\mathbb{L}(\underline{\mathbf{p}})$ denotes the rank-1 abelian group generated by the t elements $\vec{x_1}, \ldots, \vec{x_t}$ subject to relations

$$p_1\vec{x_1} \quad = \quad \cdots \quad = \quad p_t\vec{x_t} \qquad (=: \vec{c}\ ) \; .$$

Degree-shift by the dualizing element $\vec{\omega} := (t-2)\vec{c} - \sum_{i=1}^t \vec{x_i}$ serves as the Auslander-Reiten-translation τ for $\mathrm{coh}\,\mathbb{X}$; Serre-Duality $\mathrm{DExt}^1(E,F) \cong \mathrm{Hom}(F, E(\vec{\omega}))$ holds for any $E, F \in \mathrm{coh}\,\mathbb{X}$. - The representation theory of $\mathrm{coh}\,\mathbb{X}$ depends on the number $\delta_\omega := (t-2) - \displaystyle\sum_{i=1}^t \frac{1}{p_i}$; namely, $\mathrm{coh}\,\mathbb{X}$ is of *tame-domestic*, *tubular* or *wild* representation type depending on whether $\delta_\omega < 0, = 0$ or > 0.

Definition 0.1. *A sheaf* $\mathcal{T} = T_1 \oplus \cdots \oplus T_n \in \mathrm{coh}\,\mathbb{X}$*, where* T_i *are pairwise non-isomorphic indecomposable, is called a* tilting sheaf *if*

(a): $\mathrm{Ext}^1(\mathcal{T}, \mathcal{T}) = 0$ *;*

(b): $\mathcal{T}$ *generates* $\mathrm{coh}\,\mathbb{X}$ *in the sense of formation of direct sums, extensions, kernels of epimorphisms and cokernels of monomorphisms.*

Note in particular that we demand a tilting sheaf to be *multiplicity-free*. Then Condition (b) is equivalent to $n = \mathrm{rk}_{\mathbb{Z}}(\mathrm{K}_0\,\mathbb{X})$, where $\mathrm{K}_0\,\mathbb{X}$ denotes the Grothendieck-group of $\mathrm{coh}\,\mathbb{X}$ which is always free of finite rank.

Let us now define the notion of a *sink-* or *source-map* for a tilting sheaf $\mathcal{T}$ (cf. [**11**]):

DEFINITION 0.2. *Let $\mathcal{T}$ be a tilting sheaf, and F be any of its indecomposable direct summands.*

(a): *Let $N_1, \dots, N_m$ be pairwise non-isomorphic direct summands of $\mathcal{T}$, and none isomorphic to F. A morphism*

$$\rho_F : \quad F \quad \longrightarrow \quad \bigoplus_{j=1}^{m} N_j^{s_j}$$

is called a source-map for F with respect to $\mathcal{T}$ *if:*

(i): *for each $X \in \mathrm{add}(\mathcal{T} \setminus \{F\})$, each morphism from F to X factorizes over ρ_F; and*

(ii): *each endomorphism $\alpha \in \mathrm{End}(\bigoplus_{j=1}^m N_j^{s_j})$, for which $\alpha \circ f$ also satisfies the factorization property (i), is neccessarily an automorphism.*

The N_j are called immediate successors *of F in $\mathcal{T}$ with* multiplicities s_j.

(b): *Let $E_1, \dots, E_l$ be pairwise non-isomorphic direct summands of $\mathcal{T}$, and none isomorphic to F.*

A morphism

$$\sigma_F : \quad \bigoplus_{i=1}^{l} E_i^{r_i} \quad \longrightarrow \quad F$$

is called a sink-map for F with respect to $\mathcal{T}$ *if:*

(i): *for each $Y \in \mathrm{add}(\mathcal{T} \setminus \{F\})$, each morphism from Y to F factorizes over σ_F; and*

(ii): *each endomorphism $\beta \in \mathrm{End}(\bigoplus_{i=1}^l E_i^{r_i})$, for which $g \circ \beta$ also satisfies the factorization property (i), is neccessarily an automorphism.*

The E_i are called immediate predeccessors *of F in $\mathcal{T}$ with* multiplicities r_i.

Therefore property (ii) admits only modifications of sink- or source-maps by automorphisms, and demands - in rough words - a sink- or source-map to be *minimal* satisfying the appropriate factorization property. - Sink- and source-maps are thereby *uniquely* determined up to isomorphism. -

THEOREM/DEFINITION 0.3. *Let $\mathcal{T}$ be a tilting sheaf. For each arbitrary direct summand F of $\mathcal{T}$ its sink-map σ_F and its source-map ρ_F are either* both simultaneously monomorphisms *or* both simultaneously epimorphisms. *- Moreover:*

(a) *In case σ_F (and hence also ρ_F) is an epimorphism;*

$$0 \longrightarrow L \xrightarrow{\mu=\begin{bmatrix}\mu_1\\ \vdots \\ \mu_l\end{bmatrix}} \bigoplus_{i=1}^l E_i^{r_i} \xrightarrow{\sigma_F=[\sigma_1,\dots,\sigma_l]} F \longrightarrow 0\ ,$$

F is called a sink *in $\mathcal{T}$;* sink-reflection at F *(i.e. exchanging F against L) yields a new tilting sheaf $\mathcal{T}'$, where the E_i will become the immediate successors of L within $\mathcal{T}'$ (with the same "multiplicities"), and μ is a source-map for L with respect to $\mathcal{T}'$ in the sense of Definition 0.2.*

(b) *In case ρ_F (and hence also σ_F) is a monomorphism;*

$$0 \longrightarrow F \xrightarrow{\rho_F=\begin{bmatrix}\rho_1\\ \vdots \\ \rho_m\end{bmatrix}} \bigoplus_{j=1}^m N_j^{s_j} \xrightarrow{\epsilon=[\epsilon_1,\dots,\epsilon_m]} R \longrightarrow 0\ ,$$