TREE ALGEBRAS WITH NON-NEGATIVE TITS FORM

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One of the seminal results of modern representation theory of algebras is Gabriel's theorem saying that a quiver algebra $A = kQ$ accepts only finitely many isoclasses of indecomposable modules if and only if the underlying quiver of Q is a Dynkin diagram. The proof may be done by introducing the *Tits form* q_A of A and showing that the conditions above are equivalent to q_A being a positive quadratic form. In fact, the isoclasses of indecomposable A-modules are in one-to-one correspondence with the positive roots of q_A. Some years later it was shown that a quiver algebra $A = kQ$ is of tame type if and only if the corresponding Tits form is non-negative. See [8].

We recall that a finite dimensional algebra over an algebraically closed field k is said to be *tame* if for each dimension, almost all the indecomposable modules occur in a finite number of one-parameter families. We are interested in the classification of tame algebras and their representations.

Let A be a basic triangular algebra with a presentation $A = kQ/I$, where Q is a finite connected quiver without oriented cycles and I is an admissible ideal of the path algebra kQ, see [8]. Let $Q_0 = \{1, \ldots, n\}$ be the set of vertices of Q. The *Tits form* $q_A \colon \mathbb{Z}^n \to \mathbb{Z}$ is defined by

$$q_A(v) = \sum_{i=1}^{n} v(i)^2 - \sum_{i \to j} v(i)v(j) + \sum_{i,j=1}^{n} r(i,j)v(i)v(j),$$

where $i \to j$ runs over all arrows of Q and $r(i,j) = \dim_k e_j(I/IJ + JI)e_i$, where J is the ideal of kQ generated by the arrows and e_i denotes the trival path at i. An algebra A is said to be strongly simply connected if for every convex subcategory B of A the first Hochschild cohomology group $H^1(B,B)$ vanishes, see [21]. Particular examples of these algebras are the *tree algebras*, that is, algebras of the form $A = kQ/I$ where the underlying graph of Q has no cycles. It has been conjectured in [13] (see [14]) that a strongly simply connected algebra A is tame if and only if q_A is weakly non-negative, i.e. $q_A(v) \geq 0$ for vectors with non-negative coordinates (notation: $v \geq 0$). In fact, every tame algebra A has q_A weakly non-negative [14]. The conjecture has been verified for various classes of algebras, see [13]. The aim of this work is to prove the following result.

Theorem 1 *Let $A = kQ/I$ be a tree algebra of global dimension at most two and assume that q_A is non-negative. Then A is tame.*

In section 1 we prove results concerning the structure of tree algebras with a weakly non-negative Tits form accepting a positive sincere isotropic vector. We show that those algebras A have $\mathrm{g}\ell \dim A \leq 2$ and if not of polynomial growth they only accept critical subcategories of type $\tilde{\mathbb{D}}_n$. In section 2 we show a construction procedure for all tree algebras satisfying the hypothesis of the Theorem. In section 3 we present the proof of the Theorem by reducing an algebra A satisfying the hypothesis to a clannish algebra [5], using tilting theory [20] and deformations [9].

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1 Structure of tree algebras.

1.1

Let $A = kQ/I$ be a connected triangular algebra. We start by recalling some general facts and definitions. As general references, see [8, 20].

The algebra A is said to be *tame* if for every $d \in \mathbb{N}$ there is a finite family $M_1, \dots, M_s$ of $A - k[t]$-bimodules which are free finitely generated as right $k[t]$-modules and such that almost every indecomposable A-module X with $\dim_k X = d$ is isomorphic to some $M_i \otimes_{k[t]} (k[t]/(t-\lambda))$ for $1 \leq i \leq s$ and $\lambda \in k$. The minimal possible s in the definition is denoted by $\mu(d)$. A tame algebra A is said to be *domestic* (resp. of *polynomial growth*) if there is a number $m \in \mathbb{N}$ such that $\mu(d) \leq m$ (resp. $\mu(d) \leq d^m$) for every $d \in \mathbb{N}$.

Important *examples* are the following: